Propagation of Correlations in a Boltzmann Gas*

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ABSTRACT

New results are obtained on the propagation of correlations in a Boltzmann gas on the scale of the mean free path and the collisional time scale which appear to support a conjecture of M. Green's on this subject.

* A preliminary account of this work was reported at Washington, D. C.: "Convergent Binary Correlation Functions for a Boltzmann Gas", Bull. Am. Phys. Soc. 10, 531 (1965).

INTRODUCTION

In the past few years much research[1-6] has been carried out on just how the Boltzmann equation follows from the Liouville equation, and on how higher order corrections to the Boltzmann equation are found.[7-12] In the present paper these questions are re-examined using the multiple time and space scale approach[13,14] which follows from the well known Bogoliubov-Krylov technique of nonlinear mechanics.

The major new results which emerge from the present analysis follow from a careful examination of the behavior of the first-order correction to the two-particle correlation function. It is found that this function exhibits a variety of different types of behavior, among which are two different kinds of singular behavior. We have shown that by removing the secular behavior of this function, a condition on the zeroth order correlation function obtains which determines its behavior on the collisional time scale and the mean free path space scale. This leads to the verification of a conjecture of Green's[8] concerning the zeroth order correlation functions. In addition, we have found singular behavior for small relative velocities which is of an integrable kind. The corrections arising from the singular region of phase space have been shown to be of higher order than the terms kept in the present analysis.

Finally, we demonstrate that the Choh-Uhlenbeck corrections to the Boltzmann equation do drive the system to thermal equilibrium if we assume that there is no secular behavior.

I. THE BASIC EQUATIONS AND EXPANSION PROCEDURE

The starting point for the calculation is the B-B-G-K-Y hierarchy governing the reduced distribution functions f_s.

$$\left\{ \frac{\partial}{\partial t} + \sum_{i=1}^{s} \underset{\sim}{v}_i \cdot \frac{\partial}{\partial \underset{\sim}{x}_i} - \frac{1}{m} \sum_{i=1}^{s} \sum_{j=1}^{s'} \frac{\partial \phi(|\underset{\sim}{x}_{ij}|)}{\partial \underset{\sim}{x}_i} \cdot \frac{\partial}{\partial \underset{\sim}{v}_i} \right\} f_s$$
$$= \frac{n}{m} \int d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1} \sum_{i=1}^{s} \frac{\partial \phi(|\underset{\sim}{x}_{i,s+1}|)}{\partial \underset{\sim}{x}_i} \cdot \frac{\partial f_{s+1}}{\partial \underset{\sim}{v}_i}, \quad s = 1, 2, \ldots \tag{1.1}$$

where

$$\underset{\sim}{x}_{ij} = \underset{\sim}{x}_i - \underset{\sim}{x}_j \quad . \tag{1.2}$$

The notation is the same as that in Ref. 13. It is convenient to measure lengths in units of r_o, the range of the interaction potential ϕ, and to measure time in units of r_o/v_{av}, the time of a binary interaction, with v_{av} a typical particle velocity.

The Boltzmann regime is characterized by strong interactions and systems which are dilute. Thus, we introduce the expansion parameter ϵ, such that

$$n r_o^3 = \epsilon \ , \qquad \langle \phi \rangle / m v_{av}^2 \sim 1 \ , \qquad \epsilon << 1 \tag{1.3}$$

with $\langle \phi \rangle$ the characteristic strength of the potential.

Rewriting (1.1) in dimensionless units but retaining the same labels as above, we obtain

$$\left\{ \frac{\partial}{\partial t} + \sum_{i=1}^{s} \underset{\sim}{v}_i \cdot \frac{\partial}{\partial \underset{\sim}{x}_i} - \sum_{i=1}^{s} \sum_{j=1}^{s'} \frac{\partial \phi(|\underset{\sim}{x}_{ij}|)}{\partial \underset{\sim}{x}_i} \cdot \frac{\partial}{\partial \underset{\sim}{v}_i} \right\} f_s$$

$$= \epsilon \int d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1} \sum_{i=1}^{s} \frac{\partial \phi(|\underset{\sim}{x}_{i,s+1}|)}{\partial \underset{\sim}{x}_i} \cdot \frac{\partial}{\partial \underset{\sim}{v}_i} f_{s+1} . \tag{1.4}$$

In the following we will assume that the potential is repulsive and that the ensemble is spatially homogeneous. Introducing the definitions

$$\Theta(ij) = \frac{\partial \phi(|\underset{\sim}{x}_{ij}|)}{\partial \underset{\sim}{x}_i} \cdot \left(\frac{\partial}{\partial \underset{\sim}{v}_i} - \frac{\partial}{\partial \underset{\sim}{v}_j} \right) \tag{1.5}$$

and

$$H(1,\ldots,s) = \sum_{i=1}^{s} \underset{\sim}{v}_i \cdot \frac{\partial}{\partial \underset{\sim}{x}_i} - \sum_{i=1}^{s} \sum_{j>1}^{s} \Theta(ij) \tag{1.6}$$

$$\left(\frac{\partial}{\partial t} + H(1,\ldots,s) \right) f_s = \epsilon \int d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1} \sum_{i=1}^{s} \Theta(i, s+1)\, f_{s+1} . \tag{1.7}$$

Our object is to seek an asymptotic solution of (1.7) for ϵ small. We know[1] that a simple power series in ϵ does not suffice and that a more complicated asymptotic representation must be found. The multiple time and space scale procedure assumes that an expansion of the form

$$\begin{aligned} f_s = {} & f_s^{(0)}(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_s, \underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_s, t, \epsilon\underset{\sim}{x}_1,\ldots,\epsilon\underset{\sim}{x}_s, \epsilon t, \epsilon^2\underset{\sim}{x}_1,\ldots,\epsilon^2\underset{\sim}{x}_s, \epsilon^2 t, \ldots) \\ & + \epsilon f_s^{(1)}(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_s, \underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_s, t, \epsilon\underset{\sim}{x}_1,\ldots,\epsilon\underset{\sim}{x}_s, \epsilon t, \epsilon^2\underset{\sim}{x}_1,\ldots,\epsilon^2\underset{\sim}{x}_s, \epsilon^2 t, \ldots) \\ & + \epsilon^2 f_s^{(2)}(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_s, \underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_s, t, \epsilon\underset{\sim}{x}_1,\ldots,\epsilon\underset{\sim}{x}_s, \epsilon t, \epsilon^2\underset{\sim}{x}_1,\ldots,\epsilon^2\underset{\sim}{x}_s, \epsilon^2 t, \ldots) + \ldots \end{aligned} \tag{1.8}$$

will adequately represent the solution. The initial conditions on the f_s must also be expanded as in (1.8).

It is often convenient to introduce correlation functions g_s in a recursive manner

$$f_2(1\,2) = f_1(1)\,f_1(2) + g_2(1\,2)$$
$$f_3(1\,2\,3) = f_1(1)\,f_1(2)\,f_1(3) + \sum_P f_1(1)\,g_2(2\,3) + g_3(1\,2\,3) \tag{1.9}$$

etc.

It will be assumed that all the g_s vanish at $t = 0$ so that the initial state is one of complete chaos and only correlations arising from interactions will be present in the system.

II. THE LOWEST ORDER BEHAVIOR

The lowest order equations follow from (1.7) and are

$$\left\{ \frac{\partial}{\partial t} + H^{(0)}(1, \ldots, s) \right\} f_s^{(0)} = 0 \quad . \tag{2.1}$$

Thus, for $s = 1$, we find

$$\frac{\partial f_1^{(0)}}{\partial t} = 0 \tag{2.2}$$

and for $s \geq 2$

$$f_s^{(0)} = e^{-H^{(0)}(1,\ldots,s)\,t}\, f_s^{(0)}\,(t = 0,\ \epsilon t, \ldots) \quad . \tag{2.3}$$

Throughout the paper we will indicate explicitly (as in (2.3)) only those arguments of functions which need special attention drawn to them.

The operator

$$S_{-t}(1, \ldots, s) \equiv e^{-H^{(0)}(1,\ldots,s)\,t} \tag{2.4}$$

was first introduced by Bogoliubov[1] and has been extensively studied by Cohen.[11] Briefly, it replaces the phase space coordinates of the s particles by their values at time zero which are calculated using the trajectories generated by $H^{(0)}(1,\ldots,s)$.

For $s = 2$, (2.3) becomes, upon using (1.9),

$$g_2^{(0)}(t, \epsilon t, 12) = [S_{-t}(12) - 1]\ f_1^{(0)}(1, \epsilon t)\ f_1^{(0)}(2, \epsilon t)$$

$$+\ S_{-t}(12)\ g_2^{(0)}(t = 0, \epsilon t, 12) \quad . \tag{2.5}$$

We first note the appearance of the nonphysical function $g_2^{(0)}(t=0, \epsilon t, 12)$ in (2.5). Such functions are a characteristic feature of the procedure used here and arise in a characteristic way. In the present instance, in solving (2.1) on the t and $\underset{\sim}{x}_{12}$ scales, it was implicitly assumed that the variations in ϵt and $\epsilon \underset{\sim}{x}_{12}$ were negligible. The error committed is of order ϵ as long as t and $\underset{\sim}{x}_{12}$ are of order unity. The general nature of this type of expansion is such that these errors are corrected order by order. In another interpretation of this procedure[14] the arguments of the functions on the various scales are assumed to be independent variables in the strict sense. In the present paper we do not take this point of view but rather require that we be as near as possible to the physical "line", defined by $t_0, \epsilon t_1, \epsilon^2 t_2, \ldots$ with $t_o = t_1 = t_2 \cdots = t$.

We next note that (2.5) predicts a long range, finite correlation on the $\underset{\sim}{x}_{12}$ scale in certain cases, even if $g_2^{(0)}(t=0, \epsilon t)$ vanishes. This phenomenon occurs when the two particles undergo a collision in the remote past, so that $(S_{-t} - 1)$ does not vanish. We will see that this region of phase space gives rise to secular behavior in the next order. Note that in thermal equilibrium, $g_2^{(0)}(t)$ has a finite range since with $f_1^{(0)}$ Maxwellian and energy conservation, $g_2^{(0)}$ vanishes for $t \gg |\underset{\sim}{x}_{12}| / |\underset{\sim}{v}_{12}|$.

To next order (1.7) gives

$$\left\{ \frac{\partial}{\partial t} + H^{(0)}(1,\ldots,s) \right\} f_s^{(1)} + \left\{ \frac{\partial}{\partial \epsilon t} + H^{(1)}(1,\ldots,s) \right\} f_s^{(0)}$$

$$= \int d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1} \sum_{i=1}^{s} \Theta(i, s+1)\, f_{s+1}^{(0)} \tag{2.6}$$

where

$$H^{(1)}(1,\ldots,s) = \sum_{i=1}^{s} \underset{\sim}{v}_i \cdot \frac{\partial}{\partial \epsilon \underset{\sim}{x}_i} \quad . \tag{2.7}$$

In particular for $s = 1$, we have

$$\frac{\partial f_1^{(1)}}{\partial t} + \frac{\partial f_1^{(0)}}{\partial \epsilon t} = \int d\underset{\sim}{x}_2\, d\underset{\sim}{v}_2\, \Theta(12)\, f_2^{(0)}(t, \epsilon t, 12) \quad . \tag{2.8}$$

Upon integrating (2.8), we find explicit secular behavior on the t scale which is removed by requiring

$$\frac{\partial f_1^{(0)}}{\partial \epsilon t} = \int d\underset{\sim}{x}_2\, d\underset{\sim}{v}_2\, \Theta(12)\, f_2^{(0)}(\infty, \epsilon t, 12) \, . \tag{2.9}$$

Therefore we also get from (2.8)

$$\frac{\partial f_1^{(1)}}{\partial t} = \int d\underset{\sim}{x}_2\, d\underset{\sim}{v}_2\, \Theta(12)\, [\, f_2^{(0)}(t, \ldots) - f_2^{(0)}(\infty, \ldots)] \, . \tag{2.10}$$

In order to demonstrate that the decomposition of (2.8) into (2.9) and (2.10) is valid, we must show that acceptable behavior for the time development of $f_1^{(0)}$ in ϵt and $f_1^{(1)}$ in t results. In contradistinction to other treatments of this problem, we cannot in fact demonstrate this here since we have no knowledge of $g_2^{(0)}(t = 0, \epsilon t)$ and further cannot obtain any information on it in this order of approximation. We must therefore proceed to next order and, among other things, look for a determination of $g_2^{(0)}(t=0, \epsilon t)$.

III. BEHAVIOR OF $f_2^{(1)}$

In this section we examine the behavior of $f_2^{(1)}$ in those regions of phase space where $|\underset{\sim}{v}_{12}|$ is of order unity and $|\underset{\sim}{x}_{12}|$ is larger than unity (greater than r_o). It will be seen that the analysis falls naturally into examining times $t < |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$ and times $t > |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$.

We start from (2.6) which for $s = 2$ reads

$$\left(\frac{\partial}{\partial t} + H^{(0)}(12)\right) f_2^{(1)} + \left(\frac{\partial}{\partial \epsilon t} + H^{(1)}(12)\right) f_2^{(0)}$$

$$= \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 \left(\Theta(13) + \Theta(23)\right) f_3^{(0)} \quad . \tag{3.1}$$

The formal solution of (3.1) is

$$f_2^{(1)}(t) = S_{-t}(12)\, f_2^{(1)}(t=0, \epsilon t)$$

$$+ \int_0^t dt'\, S_{-(t-t')}(12) \left\{ -\left(\frac{\partial}{\partial \epsilon t} + \underset{\sim}{v}_{12} \cdot \frac{\partial}{\partial \epsilon \underset{\sim}{x}}\right) f_2^{(0)}(t', \epsilon t) \right. \tag{3.2}$$

$$\left. + \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 \left(\Theta(13) + \Theta(23)\right) f_3^{(0)}(t', \epsilon t, 123) \right\} \quad .$$

We now define

$$S_{-t}(123, i3) \equiv e^{-H(123, i3)t} \tag{3.3}$$

and

$$H(123, i3) = H^{(0)}(123) + \Theta(i3) \; . \tag{3.4}$$

Thus, using (2.3), (2.9), (3.3), and (3.4) we can rewrite (3.2) as

$$
\begin{aligned}
f_2^{(1)}(t) &= S_{-t}(12)\; f_2^{(1)}(0,\epsilon t) \\
&+ \int_0^t dt'\, S_{-(t-t')}(12) \Big\{ \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 \Big[\Theta(13) \Big((S_{-t'}(123) - S_{-t'}(123,23))\, f_1^{(0)} f_1^{(0)} f_1^{(0)} \\
&+ (S_{-t'}(123,23) - S_{-\infty}(13))\, f_1^{(0)} f_1^{(0)} f_1^{(0)} + S_{-t'}(123)\,(f_3^{(0)}(0,\epsilon t) - f_1^{(0)} f_1^{(0)} f_1^{(0)}) \\
&- S_{-\infty}(13)\; g_2^{(0)}\,(t=0,13)\; f_1^{(0)}(2) \Big) \\
&+ \Theta(23) \Big((S_{-t'}(123) - S_{-t'}(123,13))\; f_1^{(0)} f_1^{(0)} f_1^{(0)} \\
&+ (S_{-t'}(123,13) - S_{-\infty}(23)) f_1^{(0)} f_1^{(0)} f_1^{(0)} + S_{-t'}(123)\,(f_3^{(0)}(0,\epsilon t) - f_1^{(0)} f_1^{(0)} f_1^{(0)}) \\
&- S_{-\infty}(23)\; g_2^{(0)}\,(t=0,23)\; f_1^{(0)}(1) \Big) \Big] \\
&- \Big(\frac{\partial}{\partial \epsilon t} + \underset{\sim}{v}_{12} \cdot \frac{\partial}{\partial \epsilon \underset{\sim}{x}} \Big) \big(f_2^{(0)}(t',\epsilon t) - f_1^{(0)}(1)\, f_1^{(0)}(2)\big) \Big\} \; . \qquad (3.5)
\end{aligned}
$$

Equation (3.5) consists of five groups of terms which we consider separately below.

The first of these is

$$
\begin{aligned}
I_1 = \int_0^t dt'\; S_{-(t-t')}(12) \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3\; &\{ [\Theta(13)\,(S_{-t'}(123) - S_{-t'}(123,23)) \\
&+ \Theta(23)\,(S_{-t'}(123) - S_{-t'}(123,13))]\; f_1^{(0)} f_1^{(0)} f_1^{(0)} \} \; . \qquad (3.6)
\end{aligned}
$$

It is clear that the [13] and [23] terms behave similarly so that we only explicitly consider the [13] terms. First note that $\Theta(13)$ provides the restriction $|\underset{\sim}{x}_{13}| \le 1$ and that the S operators cancel if for all t', $0 \le t' \le t$,

$|\underset{\sim}{x}_{23}| > 1$. Thus, $|\underset{\sim}{x}_{23}|$ must be less than unity at some time during the integration. If particles [2] and [3] interact at time $\bar{t} < t' \leq t$ and particles [1] and [3] interact at time t', these conditions are satisfied. We denote this trajectory by [23] - [13]. Other interaction sequences such as [23] - [12] - [13] and [12] - [23] - [13] are also possible but they are shown below to yield smaller contributions to (3.6) as t becomes large.

Figure 1 shows the sequence [23] - [13]. Note from the figure that

$$\underset{\sim}{x}_{23}(t') \simeq \underset{\sim}{v}_{23}(t'-\bar{t}) + \underset{\sim}{x}_{23}(\bar{t}) \quad . \tag{3.7}$$

Thus, for the sequence to occur, $\bar{t}$ must be such that

$$|\underset{\sim}{x}_{23} - \underset{\sim}{v}_{23}(t' - \bar{t})| < 1 \tag{3.8}$$

since $\underset{\sim}{x}_{23}(\bar{t}) \sim 1$.

We picture the cone of allowed velocities in relative coordinate space in Fig. 2. From Fig. 2 we conclude that the solid angle of allowable relative velocities is

$$d\Omega \sim \frac{1}{|\underset{\sim}{x}_{23}|^2} \quad . \tag{3.9}$$

The further assumption is made that

$$\Theta(13)\,(S_{-t'}(123) - S_{-t'}(123,23))\, f_1^{(0)}(1)\, f_1^{(0)}(2) \sim 1 \tag{3.10}$$

for those values of $\underset{\sim}{x}_3$ and $\underset{\sim}{v}_3$ for which the integrand does not vanish. Thus, the contribution to I_1 is

$$\int_0^t dt'\, S_{-(t-t')}(12) \int_{|\underset{\sim}{x}_{13}|<1} d\underset{\sim}{x}_{13} \int_{d\Omega} f(\underset{\sim}{v}_3)\, d\underset{\sim}{v}_3 \quad . \tag{3.11}$$

Further,

$$\underset{\sim}{v}_3 = \underset{\sim}{v}_2 - \underset{\sim}{v}_{23} \simeq \underset{\sim}{v}_2 - |\underset{\sim}{v}_{23}|\,\hat{x}_{23}, \quad \hat{x}_{23} = \underset{\sim}{x}_{23}/|\underset{\sim}{x}_{23}| \tag{3.12}$$

and

$$\underset{\sim}{x}_{23} = -\underset{\sim}{x}_{12} + \underset{\sim}{x}_{13} \simeq -\underset{\sim}{x}_{12} \tag{3.13}$$

so that we may write (3.11) as

$$\int_0^t dt'\, S_{-(t-t')}(12)\, \frac{1}{|\underset{\sim}{x}_{12}|^2} \int f\,(\underset{\sim}{v}_2 + |\underset{\sim}{v}_{23}|\,\hat{x}_{12})\, |\underset{\sim}{v}_{23}|^2\, d|\underset{\sim}{v}_{23}| \,. \tag{3.14}$$

We note that for trajectories for which $|\underset{\sim}{x}_{12}| > 1$ for all time

$$H^{(0)}(12) = \underset{\sim}{v}_{12} \cdot \partial/\partial \underset{\sim}{x}_{12} \;.$$

The expression (3.14) then becomes

$$\int_0^t \frac{dt'}{|\underset{\sim}{x}_{12} - \underset{\sim}{v}_{12}\,(t-t')|^2} \sim \frac{1}{v_{12}^{\,2}\,t}\,, \quad t > \frac{|\underset{\sim}{x}_{12}|}{|\underset{\sim}{v}_{12}|} \tag{3.15}$$

upon assuming that the velocity integration in (3.14) yields a quantity of order unity.

When we consider the other interaction sequences such as [23] - [12] - [13], etc., it is clear that the solid angle of relative velocities is

$$d\Omega \sim \frac{1}{|\underset{\sim}{x}_{12}(t')|^2}\;\frac{1}{|\underset{\sim}{x}_{23}(\bar{t})|^2} \sim \frac{1}{|\underset{\sim}{x}_{12}(t')|^4} \tag{3.16}$$

so that a faster decay than $1/t$ will result. We conclude therefore that the dominant behavior of I_1 is a $1/t$ decay.

The next group of terms in (3.5) is

$$I_2 = \int_0^t dt'\, S_{-(t-t')}(12) \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3\, [\Theta(13)\, (S_{-t'}(123,23) - S_{-\infty}(13))$$

$$+\; \Theta(23)\, (S_{-t'}(123,13) - S_{-\infty}(23))]\; f_1^{(0)} f_1^{(0)} f_1^{(0)} \;. \tag{3.17}$$

Again we only treat the [13] terms explicitly. Introducing the change of variables

$$\underset{\sim}{x}_1{}' = \underset{\sim}{x}_1 , \quad \underset{\sim}{x}_{12} = \underset{\sim}{x}_1 - \underset{\sim}{x}_2 , \quad \underset{\sim}{x}_{13} = \underset{\sim}{x}_1 - \underset{\sim}{x}_3 \tag{3.18}$$

enables us to write

$$H^{(0)}(12) = H_r(12) + \underset{\sim}{v}_1 \cdot \left(\frac{\partial}{\partial \underset{\sim}{x}_{13}} + \frac{\partial}{\partial \underset{\sim}{x}_1{}'} \right) \tag{3.19}$$

with the relative operator given by

$$H_r(12) = \underset{\sim}{v}_{12} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{12}} - \Theta(12) . \tag{3.20}$$

We also have

$$H^{(0)}(123) = H_r(12) + H_r(13) + v_1 \cdot \frac{\partial}{\partial x_1{}'} - \Theta(23) . \tag{3.21}$$

Using the above relations we can carry out the time integration for some of the terms in (3.17) with the result that the [13] contribution to I_2 becomes

$$\int d\underset{\sim}{x}_{13}\, d\underset{\sim}{v}_3 \left[e^{-(H_r(12)+H_r(13))t} - e^{-H_r(12)t} \right] f_1^{(0)} f_1^{(0)} f_1^{(0)}$$
$$+ \int_0^t dt' \; e^{-H_r(12)(t-t')} \int d\underset{\sim}{x}_{13}\, d\underset{\sim}{v}_{13} \; \underset{\sim}{v}_{13} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{13}} \left[e^{-(H_r(12)+H_r(13))t'} \right.$$
$$\left. - e^{-H_r(13)t_\infty} \right] f_1^{(0)} f_1^{(0)} f_1^{(0)} . \tag{3.22}$$

It is clear that the first integral in (3.22) is finite. The second integral in (3.22) may be written as

$$\int_0^t dt' \; e^{-H_r(12)(t-t')} \int d\underset{\sim}{x}_{13_\perp}\, d\underset{\sim}{v}_3 \; |\underset{\sim}{v}_{13}| \left[e^{-(H_r(12)+H_r(13))t'} \right.$$
$$\left. - e^{-H_r(13)t_\infty} \right]_{\underset{\sim}{x}_{13_\parallel} = -\infty}^{\underset{\sim}{x}_{13_\parallel} = +\infty} f_1^{(0)} f_1^{(0)} f_1^{(0)} \tag{3.23}$$

The trajectories appropriate for consideration of (3.23) are shown in Fig. 3. The upper limit of the $\underset{\sim}{x}_{13_\parallel}$ integration in (3.23) yields a finite result. We see this by phase space arguments similar to those used in treating (3.6). However, if no [12] interaction occurs then the argument differs and is similar to that following (7.2). For the lower limit we note that $H_r(13)$ effectively vanishes since the [13] interaction has not yet occurred. Thus (3.23) becomes

$$-\int_0^t d\tau\ e^{-H_r(12)\tau} \int d\underset{\sim}{x}_{13_\perp}\, d\underset{\sim}{v}_{13}\, |\underset{\sim}{v}_{13}| \left(e^{-H_r(12)(t-\tau)} - 1\right) f_1^{(0)} f_1^{(0)} f_1^{(0)} \tag{3.24}$$

For $t < |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$, $H_r(12)$ commutes with $|\underset{\sim}{v}_{13}|$ and (3.24) becomes

$$-\int_0^t d\tau \int d\underset{\sim}{x}_{13_\perp}\, d\underset{\sim}{v}_{13}\, |\underset{\sim}{v}_{13}| \left(e^{-H_r(12)t} - e^{-H_r(12)\tau}\right) f_1^{(0)} f_1^{(0)} f_1^{(0)} \tag{3.25}$$

which clearly vanishes. For $t > |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$, (3.24) may be written

$$-\left\{ \int_0^{|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|} d\tau + \int_{|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|}^{t} d\tau \right\} e^{-H_r(12)\tau} \int d\underset{\sim}{x}_{12}\, d\underset{\sim}{v}_{13}\, |\underset{\sim}{v}_{13}|$$

$$\times \left(e^{-H_r(12)(t-\tau)} - 1\right) f_1^{(0)} f_1^{(0)} f_1^{(0)} \quad . \tag{3.26}$$

The first term of (3.26) is just

$$-\frac{|\underset{\sim}{x}_{12}|}{|\underset{\sim}{v}_{12}|} \int d\underset{\sim}{x}_{13}\ d\underset{\sim}{v}_{13}\ |\underset{\sim}{v}_{13}| \left(e^{-H_r t} - 1\right) f_1^{(0)} f_1^{(0)} f_1^{(0)} \tag{3.27}$$

which is spatially secular. The second term of (3.26) is

$$
- \int_{|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|}^{t} d\tau \int d\underset{\sim}{x}_{13}\, d\underset{\sim}{v}_{13}\; e^{-H_r(12)\tau} |\underset{\sim}{v}_{13}|\, e^{+H_r(12)\tau}
$$

$$
\times \left[e^{-H_r(12)t} - e^{-H_r(12)\tau} \right] f_1^{(0)} f_1^{(0)} f_1^{(0)} \qquad (3.28)
$$

which vanishes in the same manner as (3.25).

We see therefore that I_2 is finite for $t < |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$ but exhibits secular behavior for $t > |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$. It should be noted that in the examination of I_2 (and I_3 to follow) we have treated the integrals as if the limits of integration were independent of $\underset{\sim}{x}_{13}$ or $\underset{\sim}{x}_{23}$. In the appendix we show that this neglect is justified since the additional contributions arising from the limits of integration all cancel.

The next group of terms we consider is

$$
I_3 = \int_0^t dt'\; S_{-(t-t')}(12) \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3\, [\Theta(13) + \Theta(23)]\; S_{-t'}(123)
$$

$$
\times \left(f_3^{(0)}(0, \epsilon t) - f_1^{(0)}(1)\, f_1^{(0)}(2)\, f_1^{(0)}(3)\right) \quad . \qquad (3.29)
$$

We use (1.9), (3.20), and (3.21) to carry out some of the time integrations in (3.29) with the result

$$
I_3 = \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 \left[e^{-H^{(0)}(123)t} - e^{-H_r(12)t} \right]
$$

$$
\times \left(\sum_P f_1^{(0)} g_2^{(0)}(0, \epsilon t) + g_3^{(0)}(0, \epsilon t) \right)
$$

$$
+ \int_0^t dt'\; e^{-H_r(12)(t-t')} \int d\underset{\sim}{x}_{13}\, d\underset{\sim}{v}_3 \left[\underset{\sim}{v}_{13} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{13}} - \Theta(23) \right]
$$

$$
\times\; e^{-H^{(0)}(123)t'} \left(\sum_P f_1^{(0)} g_2^{(0)}(0, \epsilon t) + g_3^{(0)}(0, \epsilon t) \right). \qquad (3.30)
$$

The first integral in (3.30) is finite because both the integrand and the range of integration in $\underset{\sim}{x}_3$ are finite. The $\Theta(23)$ contribution in (3.30) is also finite because if $|\underset{\sim}{x}_{12}|$ is of order unity, the $H_r(12)$ operator can only act for a finite time. If $|\underset{\sim}{x}_{12}|$ is large, then $|\underset{\sim}{x}_{23}|$ must also be large and $\Theta(23)$ then vanishes.

In order to proceed with the analysis we are forced to make assertions on the behavior of the correlation functions and then demonstrate a posteriori that they are correct. We will assume here that $g_2^{(0)}(t=0, \epsilon t)$ is finite in the region of phase space corresponding to the two particles having already collided and is zero elsewhere. The rationale for this assumption follows from the discussion after (2.5) where we concluded that $g_2^{(0)}(t, \epsilon t)$ was finite in this range of phase space. We make a similar assertion for $g_3^{(0)}(t=0, \epsilon t)$, i.e., it is only nonzero in the region of phase space corresponding to two successive binary interactions having occurred involving all three particles.

Thus, we examine the remaining terms in (3.30)

$$\int_0^t dt' \; e^{-H_r(12)(t-t')} \int d\underset{\sim}{x}_{13}\, d\underset{\sim}{v}_3 \;\; \underset{\sim}{v}_{13} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{13}}$$

$$\times\; e^{-[H_r(12)+H_r(13)-\Theta(23)]\,t'} \tag{3.31}$$

$$\times\; [f_1^{(0)}(1)\, g_2^{(0)}(23) + f_1^{(0)}(2)\, g_2^{(0)}(13) + f_1^{(0)}(3)\, g_2^{(0)}(12) + g_3^{(0)}(123)]$$

and conclude that the terms involving integration over particle [3] must be nonsecular. This follows since over most of the range of integration they are, in fact, zero and the range in which they are nonzero is of order

$1/|\underset{\sim}{x}_{13}|^2$. The only possibly secular term in (3.31) is thus

$$\int_0^t dt' \; e^{-H_r(12)(t-t')} \int d\underset{\sim}{x}_{13_\perp} \, d\underset{\sim}{v}_3 \; |\underset{\sim}{v}_{13}|$$

$$\times \; e^{-[H_r(12)+H_r(13)-\Theta(23)]t'} \, f_1^{(0)}(3) \; g_2^{(0)}(t=0,\epsilon t,12)\Bigg|_{\underset{\sim}{x}_{13_\parallel}=-\infty}^{\underset{\sim}{x}_{13_\parallel}=+\infty} \quad . \tag{3.32}$$

The contribution from the upper limit in (3.32) vanishes since the operators **stream** the arguments of $g_2^{(0)}$ into the region of phase space for which it vanishes. This can be seen from Fig. 4 since the [13] interaction changes $\underset{\sim}{v}_1$ by a quantity of order unity and $\underset{\sim}{x}_{12}$ is also changed by order unity.

For the lower limit of (3.32) we refer to Fig. 3 and note that both the $\Theta(13)$ and $\Theta(23)$ operators vanish. Thus, $\partial/\partial\underset{\sim}{x}_{13}$ reduces to the identity operator and (3.32) becomes

$$-\int_0^t dt' \; e^{-H_r(12)(t-t')} \int d\underset{\sim}{x}_{13_\perp} \, d\underset{\sim}{v}_3 \; |\underset{\sim}{v}_{13}| \; e^{-H_r(12)t'}$$

$$\times \; f_1^{(0)}(3) \; g_2^{(0)}(12, 0, \epsilon t) \quad . \tag{3.33}$$

It is then clear that for $t < |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$, (3.33) becomes

$$-t \left(\int_{|\underset{\sim}{x}_{13}| \leq 1} d\underset{\sim}{x}_{13_\perp} d\underset{\sim}{v}_3 \; |\underset{\sim}{v}_{13}| f_1^{(0)}(3) \right) \; e^{-H_r(12)t} \; g_2^{(0)}(t=0, \epsilon t, 12) \tag{3.34}$$

and is secular.

The next group of terms is

$$I_4 = -\int_0^t dt' \, S_{-(t-t')}(12) \int d\underset{\sim}{x}_3 \, d\underset{\sim}{v}_3 \; [\Theta(13) \, S_{-\infty}(13) \, g_2^{(0)}(13) \, f_1^{(0)}(2)$$

$$+ \; \Theta(23) \, S_{-\infty}(23) \; g_2^{(0)}(23) \; f_1^{(0)}(1)] \; . \tag{3.35}$$

This expression clearly vanishes since the $S_{-\infty}$ operators project the coordinates into the region of phase space for which $g_2^{(0)}$ vanishes.

The last group of terms (3.5) may be written

$$I_5 = -\int_0^t d\tau\, S_{-\tau}(12)\left(\frac{\partial}{\partial\epsilon t} + \underset{\sim}{v}_{12}\cdot\frac{\partial}{\partial\epsilon \underset{\sim}{x}_{12}}\right)$$

$$\times\ [f_2^{(0)}(t-\tau, \epsilon t) - f_1^{(0)}(1)\, f_1^{(0)}(2)]\ . \tag{3.36}$$

For $t < |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$, the S operator commutes with $\underset{\sim}{v}_{12}$, so that (3.36) becomes

$$-\int_0^t d\tau\left(\frac{\partial}{\partial\epsilon t} + \underset{\sim}{v}_{12}\cdot\frac{\partial}{\partial\epsilon \underset{\sim}{x}_{12}}\right)(f_2^{(0)}(t, \epsilon t) - S_{-\tau}\, f_1^{(0)}(1)\, f_1^{(0)}(2))\ . \tag{3.37}$$

Further, we note that

$$S_{-\tau}\, f_1^{(0)} f_1^{(0)} = f_1^{(0)}\, f_1^{(0)},\quad 0 \le \tau \le t,\quad t < |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}| \tag{3.38}$$

so (3.36) becomes

$$-\,t\left(\frac{\partial}{\partial\epsilon t} + \underset{\sim}{v}_{12}\cdot\frac{\partial}{\partial\epsilon \underset{\sim}{x}}\right) e^{-H^{(0)}(12)t}\ g_2^{(0)}(t = 0,\ \epsilon t,\ 12) \tag{3.39}$$

which is secular.

IV. PROPAGATION OF CORRELATIONS

We now want to collect all the secular terms in $f_2^{(1)}$ and require that their sum vanish so that in fact $f_2^{(1)}$ will exhibit regular behavior. We first demonstrate that if we remove the secular behavior for $t < |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$, we will also have removed it for $t > |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$.

To prove this statement we write (3.1) in the form

$$\left(\frac{\partial}{\partial t} + H^{(0)}(12)\right) f_2^{(1)}(t) = A(t) \tag{4.1}$$

and solve this to get

$$f_2^{(1)}(t) = e^{-H^{(0)}(12)(t-t_0)} f_2^{(1)}(t_0) + \int_{t_0}^{t} dt'\, e^{-H^{(0)}(12)(t-t')} A(t') \quad . \tag{4.2}$$

The trajectories contributing to (4. 2) are shown in Fig. 5. Now choose

$$t_0 = t - \frac{|\underset{\sim}{x}_{12}|}{|\underset{\sim}{v}_{12}|} - 1 \tag{4.3}$$

and write (4, 2) as

$$f_2^{(1)}(t) = e^{-H^{(0)}(12)(1+|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|)} f_2^{(1)}\left(t - 1 - \frac{|\underset{\sim}{x}_{12}|}{|\underset{\sim}{v}_{12}|}\right)$$

$$+ \left\{ \int_{t-1-|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|}^{t+1-|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|} dt' + \int_{t+1-|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|}^{t} dt' \right\} e^{-H^{(0)}(12)(t-t')} A(t') \; . \tag{4.4}$$

The last term in (4. 4) can be transformed so that (4. 4) becomes

$$f_2^{(1)}(t) = e^{-H^{(0)}(12)(1+|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|)} f_2^{(1)}\left(t - 1 - \frac{|\underset{\sim}{x}_{12}|}{|\underset{\sim}{v}_{12}|}\right)$$

$$+ \int_{t-1-|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|}^{t+1-|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|} dt'\, e^{-H^{(0)}(12)(t-t')} A(t') + \int_{0}^{-1+|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|} d\tau \; e^{-H^{(0)}(12)\tau} A(t-\tau) \; . \tag{4.5}$$

The corresponding expression for $f_2^{(1)}$ for $t < |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$ may be written

$$f_2^{(1)}(t) = e^{-H^{(0)}(12)t} f_2^{(1)}(0) + \int_0^t d\tau \; e^{-H^{(0)}(12)\tau} A(t-\tau) \; . \tag{4.6}$$

We see that the first term of (4. 5) projects $f_2^{(1)}$ into the phase space region which does not lead to secular behavior. The second term of

(4.5) is clearly of order unity. The last term of (4.5) is the same function of $(|\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|)-1$ that the last term of (4.6) is of t. Thus if (4.6) is rid of secular behavior, (4.5) will not exhibit secularities.

Upon collecting all the secular terms for $t < |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}|$, we have

$$\left(\frac{\partial}{\partial \epsilon t} + \underset{\sim}{v}_{12} \cdot \frac{\partial}{\partial \epsilon \underset{\sim}{x}}\right) e^{-H^{(0)}(12)t}\, g_2^{(0)}(t=0,\ 12)$$
$$+ \sum_{i=1}^{2} \left(\int_{|\underset{\sim}{x}_{i3}| \le 1} d\underset{\sim}{x}_{i3\perp}\, d\underset{\sim}{v}_3\, |\underset{\sim}{v}_{i3}|\, f_1^{(0)}(3) \right) e^{-H^{(0)}(12)t} g_2^{(0)}(0,12) = 0 \qquad (4.7)$$

which describes the propagation of correlations on the collisional time scale and the mean free path length scale. The quantity

$$\nu(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \epsilon t) = \sum_{i=1}^{2} \int_{|\underset{\sim}{x}_{i3}| \le 1} d\underset{\sim}{x}_{i3\perp}\, d\underset{\sim}{v}_3\, |\underset{\sim}{v}_{i3}|\, f_1^{(0)}(3) \qquad (4.8)$$

is clearly the effective collision frequency of particles [1] and [2] with [3].

We solve (4.7) and have

$$e^{-H^{(0)}(12)t}\, g_2^{(0)}[\underset{\sim}{x}_{12}, \underset{\sim}{v}_1, \underset{\sim}{v}_2,\ t=0,\ \epsilon\underset{\sim}{x}_{12},\ \epsilon t]$$
$$= \exp\left[-\int_{\epsilon t_0}^{\epsilon t} \nu(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \epsilon t')\, d\epsilon t'\right] \qquad (4.9)$$
$$\times\, e^{-H^{(0)}(12)t}\, g_2^{(0)}(\underset{\sim}{x}_{12}, \underset{\sim}{v}_1, \underset{\sim}{v}_2,\ t=0,\ \epsilon\underset{\sim}{x}_{12} - \epsilon\underset{\sim}{v}_{12}(t-t_0),\ \epsilon t_0)$$

where ϵt_0 is an arbitrary initial time. The result (4.9) is easily generalized to

$$e^{-H^{(0)}(12)(t-\bar{t})}\, g_2^{(0)}[\underset{\sim}{x}_{12}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, \bar{t}, \epsilon \underset{\sim}{x}_{12}, \epsilon t]$$

$$= \exp[-\int_{\epsilon t_0}^{\epsilon t} \nu(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \epsilon t')\, d\epsilon t'] \tag{4.10}$$

$$\times\ e^{-H^{(0)}(12)(t-\bar{t})}\, g_2^{(0)}[\underset{\sim}{x}_{12}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, \bar{t}, \epsilon \underset{\sim}{x}_{12} - \epsilon \underset{\sim}{v}_{12}(t-t_0), \epsilon t_0]$$

with the restriction

$$(t - \bar{t}) < |\underset{\sim}{x}_{12}|/|\underset{\sim}{v}_{12}| \tag{4.11}$$

which was imposed in obtaining (4.9). In view of (4.11) we can let the $H^{(0)}(12)$ operators act in (4.10) and get

$$g_2^{(0)}[\underset{\sim}{x}_{12} - \underset{\sim}{v}_{12}(t-\bar{t}), \underset{\sim}{v}_1, \underset{\sim}{v}_2, \bar{t}, \epsilon \underset{\sim}{x}_{12}, \epsilon t]$$

$$= \exp[-\int_{\epsilon t_0}^{\epsilon t} \nu(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \epsilon t')\, d\epsilon t'] \tag{4.12}$$

$$\times\ g_2^{(0)}[\underset{\sim}{x}_{12} - \underset{\sim}{v}_{12}(t-\bar{t}), \underset{\sim}{v}_1, \underset{\sim}{v}_2, \bar{t}, \epsilon \underset{\sim}{x}_{12} - \epsilon \underset{\sim}{v}_{12}(t-t_0), \epsilon t_0] \ .$$

It is clear that we have the freedom to return to the physical "line" by choosing

$$t_0 = \bar{t} \ . \tag{4.13}$$

Thus, we simply rewrite (4.12) as

$$g_2^{(0)}[\underset{\sim}{x}_{12} - \underset{\sim}{v}_{12}(t-\bar{t}), \underset{\sim}{v}_1, \underset{\sim}{v}_2, \bar{t}, \epsilon \underset{\sim}{x}_{12}, \epsilon t]$$

$$= \exp[-\int_{\epsilon \bar{t}}^{\epsilon t} \nu(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \epsilon t')\, d\epsilon t'] \tag{4.14}$$

$$\times\ g_2^{(0)}[\underset{\sim}{x}_{12} - \underset{\sim}{v}_{12}(t-\bar{t}), \underset{\sim}{v}_1, \underset{\sim}{v}_2, \bar{t}, \epsilon \underset{\sim}{x}_{12} - \epsilon \underset{\sim}{v}_{12}(t-\bar{t}), \epsilon \bar{t}] \ .$$

Equation (4.14) can be put into a more transparent form by noting from (2.1) that

$$g_2^{(0)}\,[\underset{\sim}{x}_{12},\ \underset{\sim}{v}_1,\ \underset{\sim}{v}_2,\ t,\ \epsilon \underset{\sim}{x}_{12},\ \epsilon t]$$

$$= (e^{-H^{(0)}(12)(t-\bar{t})}-1)\ f_1^{(0)} f_1^{(0)}(2) \tag{4.15}$$

$$+\ e^{-H^{(0)}(12)(t-\bar{t})}\ g_2^{(0)}\,(\underset{\sim}{x}_{12}, \underset{\sim}{v}_1, \underset{\sim}{v}_2,\ \bar{t},\ \epsilon t,\ \epsilon \underset{\sim}{x}_{12})\quad .$$

However, with the restriction (4.11), the ff term in (4.15) vanishes while the other term on the right-hand side of (4.15) is identical to the left-hand side of (4.14). Thus, (4.14) may be written as

$$g_2^{(0)}\,(\underset{\sim}{x}_{12},\ \underset{\sim}{v}_1,\ \underset{\sim}{v}_2,\ t,\ \epsilon \underset{\sim}{x}_{12},\ \epsilon t)$$

$$= \exp[-\int_{\epsilon\bar{t}}^{\epsilon t} \nu(\epsilon t')\, d\epsilon t'\,]\ g_2^{(0)}\,[\underset{\sim}{x}_{12} - \underset{\sim}{v}_{12}(t-\bar{t}),\ \underset{\sim}{v}_1, \underset{\sim}{v}_2, \bar{t},\ \epsilon \underset{\sim}{x}_{12} - \epsilon \underset{\sim}{v}_{12}(t-\bar{t}),\ \epsilon t]. \tag{4.16}$$

We can carry this one step further by projecting $g_2^{(0)}$ into the region of phase space for which it vanishes by assumption. To accomplish this we use (4.15) in the form

$$g_2^{(0)}\,[\underset{\sim}{x}_{12} - \underset{\sim}{v}_{12}\,(t-\bar{t}),\ \underset{\sim}{v}_1, \underset{\sim}{v}_2,\ \bar{t},\ \epsilon\,(\underset{\sim}{x}_{12} - \underset{\sim}{v}_{12}\,(t-\bar{t})),\ \epsilon\bar{t}\ ]$$

$$= \left(e^{-H^{(0)}[\underset{\sim}{x}_{12} - \underset{\sim}{v}_{12}(t-\bar{t})]\,(\bar{t}-\tau)}-1\right)\ f_1^{(0)}(1)\, f_1^{(0)}(2) \tag{4.17}$$

where

$$\bar{t} - \tau\ >\ \frac{|\underset{\sim}{x}_{12} - \underset{\sim}{v}_{12}\,(t-\bar{t})\,|}{|\underset{\sim}{v}_{12}|} + O\left(\frac{1}{|\underset{\sim}{v}_{12}|}\right) \tag{4.18}$$

and we have written the arguments of $H^{(0)}(12)$ explicitly to avoid confusion.

Thus, we finally obtain

$$g_2^{(0)}(\underset{\sim}{x}_{12}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t, \epsilon t, \epsilon \underset{\sim}{x}_{12}) = \exp\left[- \int_{\epsilon \bar{t}}^{\epsilon t} \nu(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \epsilon t') \, d\epsilon t' \right]$$

$$\times \left[e^{-H^{(0)}[\underset{\sim}{x} \quad - \underset{\sim}{v} \quad (t-\bar{t})](\bar{t}-\tau)} - 1 \right] f_1^{(0)}(1, \epsilon \bar{t}) \, f_1^{(0)}(2, \epsilon \bar{t}) \, . \qquad (4.19)$$

The result (4.19) is pleasing physically since it shows that the correlation is generated by the collision of two particles and then sticks out long thin arms in phase space until a collision with a third particle occurs which causes it to decay exponentially. A conjecture of Green[8] appears to be in accord with this picture. It is interesting to note that the result (4.19) supports the Bogoliubov hypothesis that higher correlation functions become functionals of the one-particle function. We see, however, that it is not a functional of f_1 at the same time but rather there is a history dependence.

We conclude therefore that we have removed the secular behavior of $f_2^{(1)}$ subject to the assumptions made earlier on the phase space behavior of the correlation functions.

V. VERIFICATION OF PHASE SPACE ASSUMPTIONS

We now want to demonstrate that there is a consistent solution of the equations of the hierarchy which has the properties we assumed earlier for the correlation functions, subject to the initial conditions of chaos at time zero. We therefore drop the assumption that $g_s^{(0)}$ vanishes unless there have been s binary interactions and are thus forced to consider all the equations of the hierarchy jointly since, in principle, $g_{s-1}^{(0)}$ will depend on $g_s^{(0)}$, etc.

We obtain from (2.6)

$$f_s^{(1)}(t) = e^{-H^{(0)}(1,\ldots,s)(t-t_0)} f_s^{(1)}(t_0)$$

$$+ \int_{t_0}^{t} dt' \, e^{-H^{(0)}(1,\ldots,s)(t-t')} \left\{ \int_{|\underset{\sim}{x}_{i,s+1}| \le 1} d\underset{\sim}{x}_{s+1} \, d\underset{\sim}{v}_{s+1} \sum_{i=1}^{s} \Theta(i, s+1) \, f_{s+1}^{(0)}(t') \right.$$

$$\left. - \left(\frac{\partial}{\partial \epsilon t} + H^{(1)}(1,\ldots,s)\right) f_s^{(0)}(t') \right\} . \tag{5.1}$$

Our object is to remove the secular terms from (5.1), and in so doing, obtain an equation of evolution for $g_s^{(0)}$ which is free of our earlier assumption. We find it easier to solve the dynamical problem piecewise than to consider the whole problem all at once. Thus, we assume that during the time interval $t - t_0$, which is order unity but greater than unity numerically, none of the s particles is interacting. We then join such solutions across an interaction interval, τ, of order unity. The rationale follows from our earlier observations that secular behavior arises in $t - t_0$ intervals and that τ intervals contribute terms of order unity.

We now substitute (2.3) into (5.1) and write

$$(t - t_0)\left(\frac{\partial}{\partial \epsilon t} + H^{(1)}(1,\ldots,s)\right) f_s^{(0)}$$

$$= - f_s^{(1)}(t) + e^{-H^{(0)}(1,\ldots,s)(t-t_0)} f_s^{(1)}(t_0)$$

$$+ \int_{t_0}^{t} dt' \, e^{-H^{(0)}(1,\ldots,s)(t-t')} \left\{ \int_{|\underset{\sim}{x}_{i,s+1}| \le 1} d\underset{\sim}{x}_{s+1} \, d\underset{\sim}{v}_{s+1} \sum_{i=1}^{s} \Theta(i, s+1) \, f_{s+1}^{(0)}(t') \right\} . \tag{5.2}$$

Since we wish to collect only the secular terms from the right-hand side of (5.2) we will lump all the nonsecular terms into a quantity called N_s. We assume that $f_s^{(1)}(t_0)$ is not secular and that we will remove the secularities from $f_s^{(1)}(t)$. Thus the first two terms on the right-hand side of (5.2) will be put in N_s. It is convenient to introduce coordinates relative to $\underset{\sim}{x}_i$ and write

$$H^{(0)}(1,\ldots,s) \equiv H_s(i) = \underset{\sim}{v}_i \cdot \frac{\partial}{\partial \underset{\sim}{x}_i} + \sum_{j=1,\, j\neq i}^{s} \underset{\sim}{v}_{ij} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{ij}} - \sum_{i=1}^{s} \sum_{j>i}^{s} \Theta(ij) \tag{5.3}$$

and

$$H^{(0)}(1,\ldots,s+1) \equiv H_{s+1}(i) = H_s(i) + \underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} - \sum_{j=1}^{s} \Theta(j, s+1)\,. \tag{5.4}$$

Equation (5.2) may then be written

$$(t - t_0)\left(\frac{\partial}{\partial \epsilon t} + H^{(1)}(1,\ldots,s)\right) f_s^{(0)}$$

$$= \sum_{i=1}^{s} \int_{t_0}^{t} dt' \Bigg\{ e^{-H_s(i)(t-t')} \int_{|\underset{\sim}{x}_{i,\,s+1}|\leq 1} d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1}$$

$$\times \left[- H_{s+1}(i) + H_s(i) + \underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} - \sum_{j=1,\, j\neq i}^{s} \Theta(j, s+1) \right]$$

$$\times\; e^{-H_{s+1}(i)(t'-t_0)}\, f_{s+1}^{(0)}(t_0) \Bigg\} + N_s\,. \tag{5.5}$$

Note that the $\Theta(j, s+1)$ terms can be included in N_s since the time interval over which they act is limited and thus they produce no secular behavior. We then carry out some of the time integrations in (5.5) and have

$$(t-t_0)\left(\frac{\partial}{\partial \epsilon t} + H^{(1)}(1,\ldots,s)\right) f_s^{(0)} = \sum_{i=1}^{s} \int\limits_{|\underset{\sim}{x}_{i,s+1}| \leq 1} d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1}$$

$$\times \left[e^{-H_{s+1}(i)(t-t_0)} - e^{-H_s(i)(t-t_0)} \right] f_{s+1}^{(0)}(t_0)$$

$$+ \sum_{i=1}^{s} \int_{t_0}^{t} dt'\; e^{-H_s(i)(t-t')} \int\limits_{|\underset{\sim}{x}_{i,s+1}| \leq 1} d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1}\, \underset{\sim}{v}_{i,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,s+1}}$$

$$\times\; e^{-H_{s+1}(i)(t'-t_0)} f_{s+1}^{(0)}(t_0) + N_s \quad . \tag{5.6}$$

It is clear that the first group of terms on the right-hand side of (5.6) now should be put into N_s.

It is convenient to add and subtract terms i (5.6) to try to mirror the earlier calculation. We therefore write

$$(t-t_0)\left(\frac{\partial}{\partial \epsilon t} + H^{(1)}(1,\ldots,s)\right) f_s^{(0)} = \sum_{i=1}^{s} \int_{t_0}^{t} dt'\; e^{-H_s(i)(t-t')}$$

$$\times \int\limits_{|\underset{\sim}{x}_{i,s+1}| \leq 1} d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1}\; \underset{\sim}{v}_{i,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,s+1}}$$

$$\times \exp\left[-\left(H_s(i) + \underset{\sim}{v}_{i,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,s+1}} - \Theta(i,s+1) - \sum_{j=1,\, j\neq i}^{s} \Theta(j,s+1) \right)(t'-t_0) \right]$$

$$\times \left[f_1^{(0)}(s+1)\, f_1^{(0)}(i)\, f_{s-1}^{(0)}(t_0) + f_1^{(0)}(s+1)\left(f_s^{(0)}(t_0) - f_1^{(0)}(i)\, f_{s-1}^{(0)}(t_0) \right) \right.$$

$$\left. + \left(f_{s+1}^{(0)}(t_0) - f_1^{(0)}(s+1)\; f_s^{(0)}(t_0) \right) \right] + N_s. \tag{5.7}$$

We first note that, in effect, the last term in the exponential operator in (5.7) can be thrown into N_s since it is only nonzero if both the ith and jth particles interact with the s + 1st particle. In the treatment of $f_2^{(1)}$ we saw that such configurations do not yield secular behavior.

To proceed with the analysis of (5.7) we next note that since the s particles do not interact with each other, (5.3) reduces to

$$H_s(i) = \underset{\sim}{v}_i \cdot \frac{\partial}{\partial \underset{\sim}{x}_i} + \sum_{j=1,\, j\neq i}^{s} \underset{\sim}{v}_{ij} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{ij}} \quad . \tag{5.8}$$

Further, in our present coordinate system, $f_1^{(0)}(s+1)$, $f_1^{(0)}(i)$ and $f_{s-1}^{(0)}(t_0)$ are all independent of $\underset{\sim}{x}_i$. Thus, $H_s(i)$ and $\underset{\sim}{v}_{i,\,s+1} \cdot \partial/\partial \underset{\sim}{x}_{i,\,s+1} - \Theta(i, s+1)$ may be regarded as commuting operators. Therefore, the first term in the integrand of (5.7) may be written as

$$\sum_{i=1}^{s} \int_{t_0}^{t} dt' \int_{|\underset{\sim}{x}_{i,\,s+1}| \leq 1} d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1}\, \underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}}$$

$$\times \exp\left[- \left(\underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} - \Theta(i, s+1) \right) (t' - t_0) \right]$$

$$\times\, f_1^{(0)}(s+1)\, f_1^{(0)}(i)\, f_{s-1}^{(0)}(t_0) \quad . \tag{5.9}$$

From the second group of terms in (5.7) we get

$$\sum_{i=1}^{s} \int_{t_0}^{t} dt' \, e^{-H_s(i)(t-t')} \int_{|\underset{\sim}{x}_{i,s+1}| \leq 1} d\underset{\sim}{x}_{i,s+1_\perp} \, d\underset{\sim}{v}_{s+1} \, |\underset{\sim}{v}_{i,s+1}|$$

$$\times \exp\left[-\left(H_s(i) + \underset{\sim}{v}_{i,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,s+1}} - \Theta(i, s+1) \right) (t' - t_0) \right]$$

$$\times f_1^{(0)}(s+1) \left(f_s^{(0)}(t_0) - f_1^{(0)}(i) \, f_{s-1}^{(0)}(t_0) \right) \Bigg|_{\underset{\sim}{x}_{i,s+1_{\parallel}} = -\infty}^{\underset{\sim}{x}_{i,s+1_{\parallel}} = +\infty} . \qquad (5.10)$$

The upper limit in (5.10) can possibly contribute a secular term. In the lower limit we note that $\underset{\sim}{v}_{i,s+1} \cdot \partial/\partial \underset{\sim}{x}_{i,s+1} - \Theta(i, s+1)$ acts in the exponential to yield the identity operator and upon using (5.8) we can bring the $H_s(i)$ operator through. The lower limit of (5.10) can thus be written

$$\sum_{i=1}^{s} \int_{t_0}^{t} dt' \int_{|\underset{\sim}{x}_{i,s+1}| \leq 1} d\underset{\sim}{x}_{i,s+1_\perp} \, d\underset{\sim}{v}_{s+1} \, |\underset{\sim}{v}_{i,s+1}| \, f_1^{(0)}(s+1) \left(f_s^{(0)}(t) - f_1^{(0)}(i) \, f_{s-1}^{(0)}(t) \right) \Bigg|_{\underset{\sim}{x}_{i,s+1_{\parallel}} = -\infty} . \qquad (5.11)$$

It is advantageous to force this expression to resemble (5.9) and we thus insert the identity operator to get

$$\sum_{i=1}^{s} \int_{t_0}^{t} dt' \int_{|\underset{\sim}{x}_{i,s+1}| \leq 1} dx_{i,s+1_\perp} \, d\underset{\sim}{v}_{s+1} \, |\underset{\sim}{v}_{i,s+1}|$$

$$\times \exp\left[-\left(\underset{\sim}{v}_{i,s+1} \cdot \partial/\partial \underset{\sim}{x}_{i,s+1} - \Theta(i, s+1) \right) (t' - t_0) \right]$$

$$\times f_1^{(0)}(s+1) \left(f_s^{(0)}(t) - f_1^{(0)}(i) \, f_{s-1}^{(0)}(t) \right) \Bigg|_{\underset{\sim}{x}_{i,s+1_{\parallel}} = -\infty} . \qquad (5.12)$$

We can further undo the $\underset{\sim}{x}_{i,\,s+1\parallel}$ integration and rewrite (5.12) with a correction term from the upper limit. Rather than pausing to do this here we collect all the results and simply rewrite (5.7) as

$$
\begin{aligned}
&(t-t_0)\left(\frac{\partial}{\partial \epsilon t} + H^{(1)}(1,\ldots,s)\right) f_s^{(0)} \\
&\quad = \sum_{i=1}^{s} \int_{t_0}^{t} dt' \int_{|\underset{\sim}{x}_{i,\,s+1}| \leq 1} d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1}\, \underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} \\
&\quad \times \exp\left[-\left(\underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} - \Theta(i,\,s+1)\right)(t'-t_0)\right] f_1^{(0)}(s+1)\, f_s^{(0)}(t) \\
&\qquad + R_s + N_s
\end{aligned}
\tag{5.13}
$$

where R_s is

$$
\begin{aligned}
R_s = \sum_{i=1}^{s} \int_{t_0}^{t} dt'\; e^{-H_s(i)(t-t')} \Bigg\{ &\int d\underset{\sim}{x}_{i,\,s+1}\, d\underset{\sim}{v}_{s+1}\, |\underset{\sim}{v}_{i,\,s+1}| \\
&\times \exp\left[-\left(H_s(i) + \underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} - \Theta(i,\,s+1)\right)(t'-t_0)\right] \\
&\times f_1^{(0)}(s+1)\left(f_s^{(0)}(t_0) - f_1^{(0)}(i)\, f_{s-1}^{(0)}(t_0)\right)\Bigg|_{\underset{\sim}{x}_{i,\,s+1\parallel}}^{=+\infty} \\
&+ \int d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1}\, \underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} \\
&\times \exp\left[-\left(H_s(i) + \underset{\sim}{v}_{i} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} - \Theta(i,\,s+1)\right)(t'-t_0)\right] \\
&\times \left(f_{s+1}^{(0)}(t_0) - f_1^{(0)}(s+1)\, f_s^{(0)}(t_0)\right)\Bigg\}
\end{aligned}
$$

This equation continued on next page.

$$-\sum_{i=1}^{s} \int_{t_0}^{t} dt' \int d\underset{\sim}{x}_{i,\,s+1_\perp}\, d\underset{\sim}{v}_{s+1}\, |\underset{\sim}{v}_{i,\,s+1}|$$

$$\times \exp\left[-\left(\underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} - \Theta(i, s+1)\right)(t' - t_0)\right]$$

$$f_1^{(0)}(s+1)\left(f_s^{(0)}(t) - f_1^{(0)}(i)\, f_{s-1}^{(0)}(t)\right)\Bigg|^{\underset{\sim}{x}_{i,\,s+1_{\parallel}} = +\infty} \quad . \tag{5.14}$$

Next we note that the operator

$$\exp\left[-\left(\underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} - \Theta(i, s+1)\right)(t'-t_0)\right] \tag{5.15}$$

occurring in (5.13) describes the usual two-particle interaction and for

$$t' - t_0 > 1 \tag{5.16}$$

we can replace (5.15) by

$$\exp\left[-\left(\underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} - \Theta(i, s+1)\right) t_\infty\right] \quad . \tag{5.17}$$

Thus, we may rewrite (5.13) as

$$\left(\frac{\partial}{\partial \epsilon t} + H^{(1)}(1, \ldots, s)\right) f_s^{(0)}$$

$$= \sum_{i=1}^{s} \int d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1}\, \underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}}$$

$$\times \exp\left[-\left(\underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} - \Theta(i, s+1)\right) t_\infty\right] f_1^{(0)}(s+1)\, f_s^{(0)}(t)$$

$$+ \quad \frac{R_s + N_s}{(t - t_0)} \quad . \tag{5.18}$$

If we now substitute the cluster expansion in (5.18) there results

$$\left(\frac{\partial}{\partial \epsilon t} + H^{(1)}(1,\ldots,s)\right) g_s^{(0)}$$

$$= \sum_{i=1}^{s} \int d\underset{\sim}{x}_{s+1}\, d\underset{\sim}{v}_{s+1}\, \underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}}$$

$$\times \exp\left[-\left(\underset{\sim}{v}_{i,\,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,\,s+1}} - \Theta(i,s+1)\right) t_\infty\right] f_1^{(0)}(s+1)\, g_s^{(0)}(t)$$

$$+ F_s\{R_s\} + G_s\{N_s\} \tag{5.19}$$

where F_s and G_s are functionals of the R_s and N_s, respectively. We do not write out the explicit expressions for F_s and G_s here but simply note that they are complicated sums of products of the R's and N's involving j particles with $1 \leq j \leq s$.

From (2.1) we have, for the time interval $t - t_0$,

$$\left(\frac{\partial}{\partial t} + H^{(0)}(1,\ldots,s)\right) g_s^{(0)} = 0 . \tag{5.20}$$

Thus, combining (5.19) and (5.20) we get

$$\frac{\partial g_s^{(0)}}{\partial t} + H^{(0)}(1,\ldots,s)\, g_s^{(0)} + \epsilon\, \frac{\partial g_s^{(0)}}{\partial \epsilon t} + \epsilon H^{(1)}(1,\ldots,s)\, g_s^{(0)}$$

$$= -\, \epsilon\, \nu\, (\underset{\sim}{v}_1 \ldots \underset{\sim}{v}_s, \epsilon t)\, g_s^{(0)} + \epsilon\, F'\{R_s\} + \epsilon\, G\{N_s\} . \tag{5.21}$$

In writing (5.21) we have defined the generalized collision frequency

$$\nu(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_s, \epsilon t) = \sum_{i=1}^{s} \int d\underset{\sim}{x}_{s+1_\perp} \, d\underset{\sim}{v}_{s+1} \, |\underset{\sim}{v}_{i,s+1}|$$

$$\exp\left[- \left(\underset{\sim}{v}_{i,s+1} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{i,s+1}} - \Theta(i, s+1) \right) t_\infty \right] f_1^{(0)}(s+1) \Bigg|_{x_{i,s+1_\parallel} = -\infty} . \tag{5.22}$$

The expression on the right-hand side of (5.22) evaluated at $x_{i,\, s+1_{\parallel}} = +\infty$ has been lumped with F to become F'.

The piecing procedure for solving the complete problem then consists of using (5.21) in the intervals in which the s particles are not interacting,and then using (2.1) in the intervals in which they do interact. Clearly, for a finite number of interactions we introduce a relative error of order ϵ in this way since we have used (2.1) rather than an equation good to one higher order in ϵ as is (5.21). This relative error does not influence the outcome of the proof, however, since we wish to make statements concerning a zeroth order quantity.

In addition to the piecing procedure we will use an iterative procedure to solve (5.21). The lowest order approximation consists of neglecting $F'\{R_s\}$ in (5.21). Note that the interval can be as long as $t - t_0 \sim 1/\epsilon$. Now, $G\{N_s\}$ certainly has at least one term which behaves as $1/t - t_0$ while all the others behave as

$$\frac{1}{(t-t_0)^p} , \quad 1 < p \leq s . \tag{5.23}$$

Therefore, the $G\{N_s\}$ terms can at best produce a term which is of order ϵ in $(g_s^{(0)})_0$ after integrating for a time of order $1/\epsilon$. We proceed to the next iterate by substituting $(g_s^{(0)})_0$ into $F'\{R_s\}$. By carefully examining R_s in (5.14) using arguments analogous to those used earlier for $f_2^{(1)}$ we can conclude that no secular terms are produced. Thus, $F'\{R_s\}$ actually behaves in the same fashion as the $G\{N_s\}$ terms did in the lowest approximation. We therefore see that the equations governing $(g_s^{(0)})_1$ will essentially be identical to the equations for $(g_s^{(0)})_0$. The net effect is, then, that as far as the present proof is concerned, we can simply drop the F' and G terms from consideration and we need not iterate at all.

We can now return to (5.21) and for $s = 2$, we have

$$\frac{\partial g_2^{(0)}}{\partial t} + H^{(0)}(1\,2)\, g_2^{(0)} + \epsilon\, \frac{\partial g_2^{(0)}}{\partial \epsilon t} + \epsilon H^{(1)}(1\,2)\, g_2^{(0)}$$

$$= -\ \epsilon \nu\, (\underset{\sim}{v}_1, \underset{\sim}{v}_2, \epsilon t)\ g_2^{(0)} \quad . \tag{5.24}$$

Note now that (5.24) holds in all regions of phase space and for time intervals in which particles [1] and [2] do not interact with each other. However, we can now apply (5.24) to the situation in which

$$g_2^{(0)}\ (t = 0,\ \ \epsilon t = 0,\ \ 1, 2) \ = \ 0 \tag{5.25}$$

and we see that it remains zero until an interaction takes place. During the time of interaction we use

$$\frac{\partial g_2^{(0)}}{\partial t} + H^{(0)}(12)\ g_2^{(0)} = -\ \Theta\,(12)\ f_1^{(0)} f_1^{(0)} \tag{5.26}$$

which then creates the correlation.

Thus, the above arguments have related $g_2^{(0)}$ to its physical initial value and have demonstrated that the assumption that $g_2^{(0)}$ vanishes in the region of phase space corresponding to no previous interaction is, in fact, correct. As a by-product we have also obtained (5.21) for the s particle correlations, and can thus describe the long time, long space behavior of all the correlations.

VI. BEHAVIOR OF $f_2^{(1)}$ for $|\underset{\sim}{v}_{12}| << 1$

On physical grounds we are led to suspect that some sort of singular behavior arises for $|\underset{\sim}{v}_{12}| << 1$ since two particles might then stay together for times comparable to the time between collisions. We therefore investigate here the nature of this singular behavior. We assume that $\underset{\sim}{v}_{12}$ is small and that the range in configuration space is limited by

$$\phi(|\underset{\sim}{x}_{12}|) \leq \frac{v_{12}^2}{2} \quad . \tag{6.1}$$

A crude estimate of the behavior of $f_2^{(1)}$ follows from (3.2) under these assumptions. We may assume that the integrand of the t' integration is of order unity for

$$|e^{-H^{(0)}(12)(t-t')} \underset{\sim}{x}_{12}| \sim 1 \quad . \tag{6.2}$$

For such configurations, the interval in t' for which this is so is of order $|\underset{\sim}{v}_{12}|^{-1}$ which indicates that $f_2^{(1)}$ varies as $|\underset{\sim}{v}_{12}|^{-1}$ for $|\underset{\sim}{v}_{12}| << 1$.

To treat this more formally we scale the variables in the usual manner of asymptotic analysis. We write

$$\underset{\sim}{v}_{12} = \epsilon^{\delta} \, \underset{\sim}{\tilde{v}}_{12} \, , \quad \underset{\sim}{\tilde{v}}_{12} \sim 1 \quad . \tag{6.3}$$

Assuming that (6.1) holds, we have

$$\phi\,(|\underset{\sim}{x}_{12}|) \sim \epsilon^{2\delta}\,\tilde{\phi}(|\underset{\sim}{x}_{12}|),\ \tilde{\phi}(|\underset{\sim}{x}_{12}|) \sim 1\,. \tag{6.4}$$

We then write (2.1) for $s = 2$ in the form

$$\left\{ \frac{\partial}{\partial t} + \underset{\sim}{v}_{12} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{12}} - \frac{\partial \phi}{\partial \underset{\sim}{x}_{12}} \cdot \left(\frac{\partial}{\partial \underset{\sim}{v}_1} - \frac{\partial}{\partial \underset{\sim}{v}_2} \right) \right\} g_2^{(0)}$$

$$= \frac{\partial \phi}{\partial \underset{\sim}{x}_{12}} \cdot \left(\frac{\partial}{\partial \underset{\sim}{v}_1} - \frac{\partial}{\partial \underset{\sim}{v}_2} \right) f_1^{(0)}(1)\, f_1^{(0)}(2) \quad . \tag{6.5}$$

Introducing the variables

$$\underset{\sim}{v}_{12} = \underset{\sim}{v}_1 - \underset{\sim}{v}_2\ , \quad \underset{\sim}{V} = \underset{\sim}{v}_1 + \underset{\sim}{v}_2 \tag{6.6}$$

enables us to write

$$\frac{\partial}{\partial \underset{\sim}{v}_1} - \frac{\partial}{\partial \underset{\sim}{v}_2} = 2\, \frac{\partial}{\partial \underset{\sim}{v}_{12}} \tag{6.7}$$

and, by Taylor expanding,

$$f_1^{(0)} \begin{pmatrix} \underset{\sim}{v}_1 \\ \underset{\sim}{v}_2 \end{pmatrix} = f_1^{(0)} \left(\frac{\underset{\sim}{V}}{2} \right) \pm \frac{\underset{\sim}{v}_{12}}{2} \cdot \nabla_V\, f_1^{(0)} + \frac{\underset{\sim}{v}_{12}\underset{\sim}{v}_{12} : \nabla_V \nabla_V f_1^{(0)}}{8} + \ldots \tag{6.8}$$

We now rewrite (6.5) in the scaled variables and require that all the terms remain in the scaled version. This yields

$$\left\{ \frac{\partial}{\partial \tilde{t}} + \tilde{\underset{\sim}{v}}_{12} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{12}} - 2\, \frac{\partial \tilde{\phi}}{\partial \underset{\sim}{x}_{12}} \cdot \frac{\partial}{\partial \tilde{\underset{\sim}{v}}_{12}} \right\} \tilde{g}_2^{(0)}$$

$$= \frac{1}{2}\, \frac{\partial \tilde{\phi}}{\partial \underset{\sim}{x}_{12}} \cdot \frac{\partial}{\partial \underset{\sim}{v}_{12}} \left\{ f_1^{(0)} \left(\frac{\underset{\sim}{V}}{2} \right) \tilde{\underset{\sim}{v}}_{12}\, \tilde{\underset{\sim}{v}}_{12} : \nabla_V \nabla_V\, f_1^{(0)} \right.$$

$$\left. - (\tilde{\underset{\sim}{v}}_{12} \cdot \nabla_V\, f_1^{(0)})^2 \right\} \tag{6.9}$$

with the additional scaling

$$g_2^{(0)} = \epsilon^{2\delta}\, \tilde{g}_2^{(0)} \tag{6.10}$$

and

$$\frac{d\tilde{t}}{dt} = \epsilon^{\delta} . \tag{6.11}$$

We next turn to (3.1) and write it in the form

$$\left\{ \frac{\partial}{\partial t} + \underset{\sim}{v}_{12} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{12}} - 2 \frac{\partial \phi}{\partial \underset{\sim}{x}_{12}} \cdot \frac{\partial}{\partial \underset{\sim}{v}_{12}} \right\} f_2^{(1)}$$

$$= H\,(f_3^{(0)}) - \frac{\partial f_2^{(0)}}{\partial \epsilon t} - \underset{\sim}{v}_{12} \cdot \frac{\partial f_2^{(0)}}{\partial \epsilon \underset{\sim}{x}_{12}} . \tag{6.12}$$

We note that the second term on the right-hand side of (6.12) is clearly of order unity, while $H(f_3^{(0)})$ is at best of order unity. The third term is

$$\underset{\sim}{v}_{12} \cdot \frac{\partial}{\partial \epsilon \underset{\sim}{x}} f_2^{(0)} = \epsilon^{3\delta} \, \underset{\sim}{\tilde{v}}_{12} \cdot \frac{\partial \tilde{g}_2^{(0)}}{\partial \epsilon \underset{\sim}{x}} . \tag{6.13}$$

The scaled version of (6.12) is then

$$\epsilon^{\delta} \left(\frac{\partial}{\partial \tilde{t}} + \underset{\sim}{\tilde{v}}_{12} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{12}} - 2 \frac{\partial \tilde{\phi}}{\partial \underset{\sim}{x}_{12}} \cdot \frac{\partial}{\partial \underset{\sim}{\tilde{v}}_{12}} \right) f_2^{(1)} = O(1) \tag{6.14}$$

which indicates that

$$f_2^{(1)} = \tilde{f}_2^{(1)} / \epsilon^{\delta} \tag{6.15}$$

and verifies the crude estimate of the behavior of $f_2^{(1)}$.

We next examine the question of whether, for some value of δ, the ϵ expansion used in the paper becomes invalid. The ratio

$$\frac{\epsilon f_2^{(1)}}{g_2^{(0)}} \sim \epsilon^{1-3\delta} \tag{6.16}$$

becomes of order unity for $\delta = 1/3$. We then must examine the contribution to f_1 arising from $f_2^{(1)}$ and compare it with the contribution from $g_2^{(0)}$.

The contribution is

$$\int_{v_{12}=0}^{v_{12}\sim\epsilon^{1/2}} d\underset{\sim}{v}_{12} \int_{\phi(|\underset{\sim}{x}_{12}|)\le v_{12}^2} d\underset{\sim}{x}_{12} \quad \frac{\partial\phi}{\partial \underset{\sim}{x}_{12}} \cdot \left(\frac{\partial}{\partial \underset{\sim}{v}_1} - \frac{\partial}{\partial \underset{\sim}{v}_2}\right) f_2^{(1)} . \qquad (6.17)$$

The spatial volume of integration is of order unity for $\underset{\sim}{x}_{12}$ determined by (6.1). Further,

$$\frac{\partial\phi}{\partial \underset{\sim}{x}_{12}} \cdot \left(\frac{\partial}{\partial \underset{\sim}{v}_1} - \frac{\partial}{\partial \underset{\sim}{v}_2}\right) \sim 2 \; \frac{\partial\tilde{\phi}}{\partial \underset{\sim}{x}_{12}} \cdot \frac{\partial \tilde{f}_2^{(1)}}{\partial \tilde{\underset{\sim}{v}}_{12}} \sim 1 \quad . \qquad (6.18)$$

Thus, the integral in (6.17) is of order

$$\int_{v_{12}=0}^{v_{12}\sim\epsilon^{1/3}} |v_{12}|^2 \, d\underset{\sim}{v}_{12} \sim \epsilon . \qquad (6.19)$$

We therefore conclude that although $f_2^{(1)}$ diverges for $|\underset{\sim}{v}_{12}| << 1$, the contribution to the kinetic equation for f_1 is of higher order in ϵ and, thus, the original ordering is not upset for $\delta \geq 1/3$.

VII. THE BOLTZMANN AND CHOH-UHLENBECK RESULTS

We can now go back and point out the connection between the present work and the Boltzmann and Choh-Uhlenbeck results.

To this end, we first go back to (2.9) and (2.10) and write them as

$$\frac{\partial f_1^{(0)}}{\partial \epsilon t} = \int d\underset{\sim}{x}_2 \, d\underset{\sim}{v}_2 \;\; \Theta(12) \;\; S_{-\infty}(12) \; [\, f_1^{(0)} f_1^{(0)} + g_2^{(0)}(0, \epsilon t, 12)] \qquad (7.1)$$

and

$$\frac{\partial f_1^{(1)}}{\partial t} = \int d\underset{\sim}{x}_2\, d\underset{\sim}{v}_2\, \Theta(12)\, (S_{-t} - S_{-\infty})\, [f_1^{(0)} f_1^{(0)} + g_2^{(0)}(t=0, \epsilon t, 12)] \,. \tag{7.2}$$

In considering (7.1) we immediately note that the $S_{-\infty}$ operator projects the arguments of $g_2^{(0)}$ into the phase space region in which it vanishes. As for the ff term in (7.1), we note that Bogoliubov[1] has demonstrated that this is just another version of the standard Boltzmann collision integral. The H theorem then guarantees the behavior of $f_1^{(0)}$.

We next turn to (7.2). The $\Theta(12)$ operator restricts $\underset{\sim}{x}_{12}$ to be of order unity at time t or the integral will vanish. Trajectories which have $\underset{\sim}{x}_{12}(t) \sim 1$ at time t and $\underset{\sim}{x}_{12}(0) \sim 1$ are the only ones which will contribute to the integral. Thus, we require

$$|\underset{\sim}{x}_{12}(t) - \underset{\sim}{x}_{12}(0)| < 1 \tag{7.3}$$

or, estimating,

$$|\underset{\sim}{v}_{12}|\, t < 1 \,. \tag{7.4}$$

We can write (7.2) as

$$\frac{\partial f_1^{(1)}}{\partial t} = \frac{\partial}{\partial \underset{\sim}{v}_1} \cdot \int d\underset{\sim}{v}_{12}\, \underset{\sim}{F}(\underset{\sim}{v}_{12}) \tag{7.5}$$

where the vector $\underset{\sim}{F}$ vanishes if $|\underset{\sim}{v}_{12}| > 1/t$ and is at most of order unity if $|\underset{\sim}{v}_{12}| < 1/t$. Thus,

$$\frac{\partial f_1^{(1)}}{\partial t} \sim \frac{1}{t^3} \tag{7.6}$$

and this contribution is well behaved as t becomes large.

We have then justified the original decomposition of (2.8) into (2.9) and (2.10) and have demonstrated that the standard Boltzmann result obtains.

The next order result is, from (1.7),

$$\frac{\partial f_1^{(2)}}{\partial t} + \frac{\partial f_1^{(1)}}{\partial \epsilon t} + \frac{\partial f_1^{(0)}}{\partial \epsilon^2 t} = \int d\underset{\sim}{x}_2\, d\underset{\sim}{v}_2\ \Theta(12)\ f_2^{(1)}(t, \epsilon t, \epsilon^2 t)\ . \tag{7.7}$$

Upon removing secular behavior, we get

$$\frac{\partial f_1^{(1)}}{\partial \epsilon t}(\infty, \epsilon t, \epsilon^2 t) + \frac{\partial f_1^{(0)}}{\partial \epsilon^2 t}(\epsilon t, \epsilon^2 t) = \int d\underset{\sim}{x}_2 d\underset{\sim}{v}_2\ \Theta(12)\ f_2^{(1)}(\infty, \epsilon t, \epsilon^2 t) \tag{7.8}$$

and

$$\frac{\partial f_1^{(2)}}{\partial t} + \frac{\partial f_1^{(1)}}{\partial \epsilon t}(t, \epsilon t, \epsilon^2 t) - \frac{\partial f_1^{(1)}}{\partial \epsilon t}(\infty, \epsilon t, \epsilon^2 t)$$

$$= \int d\underset{\sim}{x}_2\, d\underset{\sim}{v}_2\ \Theta(12)\ [f_2^{(1)}(t, \epsilon t, \epsilon^2 t) - f_2^{(1)}(\infty, \epsilon t, \epsilon^2 t)]\ . \tag{7.9}$$

We, of course, expect that (7.8) will provide corrections to the Boltzmann equation and that (7.9) will describe the quick approach to the collisional time scale.

From (3.1), we get

$$f_2^{(1)}(t) = \int_{t_0}^{t} dt'\ e^{-H^{(0)}(12)(t-t')} \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3\ [\Theta(13) + \Theta(23)]\, e^{-H^{(0)}(123)(t'-t_0)} f_3^{(0)}$$

$$- e^{-H^{(0)}(12)(t-t_0)} \int_{t_0}^{t} dt' \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3\, \Theta(13)\, e^{-H^{(0)}(13)t_\infty} f_1^{(0)} f_1^{(0)} f_1^{(0)}$$

This equation continued on next page.

$$- e^{-H^{(0)}(12)(t-t_0)} \int_{t_0}^{t} dt' \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3\, \Theta(23)\, e^{-H^{(0)}(23)t_\infty} f_1^{(0)} f_1^{(0)} f_1^{(0)}$$

$$+ e^{-H^{(0)}(12)(t-t_0)} f_2^{(1)}(t_0) - \int_{t_0}^{t} dt'\, e^{-H^{(0)}(12)(t-t')} \underset{\sim}{v}_{12} \cdot \frac{\partial}{\partial \epsilon \underset{\sim}{x}} f_2^{(0)}(t') . \tag{7.10}$$

In writing (7.10) we have used (7.1) and the fact that the $S_{-\infty}$ operator projects $g_2^{(0)}$ into the region in which it vanishes.

For substitution into (7.8) we need $f_2^{(1)}$ for $|\underset{\sim}{x}_{12}| \leq 1$ and for t large. We note that $f_2^{(1)}$ itself cannot possibly be secular in this range of phase space since $\underset{\sim}{x}_{12}$ cannot possibly be parallel to $\underset{\sim}{v}_{12}$. However, contributions to $f_2^{(1)}$ can arise from correlation functions whose arguments are such that $\underset{\sim}{x}_{12}$ was parallel to $\underset{\sim}{v}_{12}$ at some time. In principle, we then should use the correlations corrected for the exponential behavior that we found earlier. This is one of the essential differences between the present treatment and earlier treatments of this problem.

The last term on the right-hand side of (7.10) then vanishes since the $\epsilon\underset{\sim}{x}$ derivative of $g_2^{(0)}$ is only nonzero if $\underset{\sim}{x}_{12}$ is parallel to $\underset{\sim}{v}_{12}$. We can now rewrite (7.10) in the form

$$f_2^{(1)} = \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 \left(e^{-H^{(0)}(123)(t-t_0)} - e^{-H^{(0)}(12)(t-t_0)} \right) f_1^{(0)} f_1^{(0)} f_1^{(0)}$$

$$+ \int_{t_0}^{t} dt'\, e^{-H^{(0)}(12)(t-t')} \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3\, \underset{\sim}{v}_3 \cdot \frac{\partial}{\partial \underset{\sim}{x}_3}\, e^{-H^{(0)}(123)(t'-t_0)} f_1^{(0)} f_1^{(0)} f_1^{(0)}$$

$$+ \int_{t_0}^{t} dt'\, e^{-H^{(0)}(12)(t-t')} \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \left(\Theta(13) + \Theta(23) \right) e^{-H^{(0)}(123)(t'-t_0)}$$

This equation continued on next page.

$$\times \left[f_1^{(0)}(3)\, g_2^{(0)}(12) + f_1^{(0)}(2)\, g_2^{(0)}(13) + f_1^{(0)}(1)\, g_2^{(0)}(23) + g_3^{(0)}(123) \right]$$

$$- \int_{t_0}^{t} dt' \; e^{-H^{(0)}(12)(t-t_0)} \left\{ \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \; \Theta(13) \left[\left(e^{-H^{(0)}(13)(t_\infty - t_0)} - e^{-H^{(0)}(13)(t'-t_0)} \right) \right.\right.$$

$$\left. + \; e^{-H^{(0)}(13)(t'-t_0)} \right] \; f_1^{(0)} f_1^{(0)} f_1^{(0)}$$

$$+ \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 \; \Theta(23) \left[\left(e^{-H^{(0)}(23)(t_\infty - t_0)} - e^{-H^{(0)}(23)(t'-t_0)} \right) \right.$$

$$\left.\left. + \; e^{-H^{(0)}(23)(t'-t_0)} \right] \; f_1^{(0)} f_1^{(0)} f_1^{(0)} \right\}$$

$$+ \; e^{-H^{(0)}(12)(t-t_0)} \; f_2^{(1)}(t_0) \quad . \qquad (7.11)$$

The second term on the right-hand side of (7.11) vanishes under the assumption that t is less than $|\underset{\sim}{x}_3|/|\underset{\sim}{v}_3|$ since the upper and lower limits of the $\underset{\sim}{x}_{3_{\parallel}}$ integration are then identical.

We next consider the term

$$- \int_{t_0}^{t} dt' \; e^{-H^{(0)}(12)(t-t_0)} \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 \; \Theta(13) \; e^{-H^{(0)}(13)(t'-t_0)} \, f_1^{(0)} f_1^{(0)} f_1^{(0)} \qquad (7.12)$$

which can be written

$$\int_{t_0}^{t} dt' \; e^{-H^{(0)}(12)(t-t_0)} \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \; H^{(0)}(13) \; e^{-H^{(0)}(13)(t'-t_0)} \, f_1^{(0)} f_1^{(0)} f_1^{(0)} . \qquad (7.13)$$

The term in the $\underset{\sim}{x}_{13_{\parallel}}$ integration which would add to (7.13) in obtaining it from (7.12) vanishes for the same reason as given above after (7.11). We can now integrate (7.13) and get

$$\int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 \left(e^{-\overset{\times}{H}^{(0)}(12)(t-t_0)} - e^{-H^{(0)}(12)(t-t_0)}\, e^{-H^{(0)}(13)(t-t_0)}\right) f_1^{(0)} f_1^{(0)} f_1^{(0)} . \tag{7.14}$$

Clearly, the terms involving particles [2] and [3] can be handled in the same way. Thus, (7.11) becomes

$$\begin{aligned}
f_2^{(1)} = \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 & \Big[e^{-H^{(0)}(123)(t-t_0)} + e^{-H^{(0)}(12)(t-t_0)} \\
& - e^{-H^{(0)}(12)(t-t_0)}\, e^{-H^{(0)}(13)(t-t_0)} - e^{-H^{(0)}(12)(t-t_0)}\, e^{-H^{(0)}(23)(t-t_0)} \Big] f_1^{(0)} f_1^{(0)} f_1^{(0)} \\
& - \int_{t_0}^{t} dt'\, e^{-H^{(0)}(12)(t-t_0)} \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 \Big[\Theta(13) \left(e^{-H^{(0)}(13)(t_\infty - t_0)} - e^{-H^{(0)}(13)(t'-t_0)} \right) \\
& + \Theta(23) \left(e^{-H^{(0)}(23)(t_\infty - t_0)} - e^{-H^{(0)}(23)(t'-t_0)} \right) \Big] f_1^{(0)} f_1^{(0)} f_1^{(0)} \\
& + \int_{t_0}^{t} dt'\, e^{-H^{(0)}(12)(t-t')} \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 \left(\Theta(13) + \Theta(23) \right) e^{-H^{(0)}(123)(t'-t_0)} \\
& \times \Big[f_1^{(0)}(3)\, g_2^{(0)}(12) + f_1^{(0)}(2)\, g_2^{(0)}(13) + f_1^{(0)}(1)\, g_2^{(0)}(23) + g_3^{(0)}(123) \Big] \\
& + e^{-H^{(0)}(12)(t-t_0)} \Big[f_1^{(0)}(1)\, f_1^{(1)}(t_0, \epsilon t, 2) + f_1^{(0)}(2)\, f_1^{(1)}(t_0, \epsilon t, 1) + g_2^{(1)}(t_0) \Big] .
\end{aligned} \tag{7.15}$$

Now consider the contribution to (7.15) arising from

$$\int_{t_0}^{t} dt'\, e^{-H^{(0)}(12)(t-t')} \int d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3 \left(\Theta(13) + \Theta(23) \right) e^{-H^{(0)}(123)(t'-t_0)} \times f_1^{(0)}(3)\; g_2^{(0)}(12) . \tag{7.16}$$

A typical set of trajectories contributing to (7.16) is shown in Fig. 6. The same figure with [1] and [2] interchanged is also appropriate. Note that there is a [12] interaction prior to t_0 and there is thus a nonzero value for $g_2^{(0)}(12)$. Further, there is a [13] or [23] interaction as required by the presence of $\Theta(13) + \Theta(23)$. Finally $|\underset{\sim}{x}_{12}| \leq 1$ at time t. We note that the distance $\overline{BC}$ is of order $\underset{\sim}{v}_1(t-t_0) \sim \overline{AC}$. Thus, assuming the integrand of the phase space integration to be of order unity, the integral itself will be of order $1/|\underset{\sim}{v}_1|^2(t-t_0)^2$ because of the solid angle. The further t' integration then yields the behavior $1/t-t_0$. It is clear that if a greater number of binary interactions had occurred in the interval $t - t_0$ which still satisfied the above restrictions, a result would have been obtained which vanished more rapidly than $(t-t_0)^{-1}$.

The terms

$$\int_{t_0}^{t} dt' \; e^{-H^{(0)}(12)(t-t')} \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \left[\Theta(13) \; e^{-H^{(0)}(123)(t'-t_0)} f_1^{(0)}(1) \, g_2^{(0)}(23) + \Theta(23) \; e^{-H^{(0)}(123)(t'-t_0)} f_1^{(0)}(2) \; g_2^{(0)}(13)\right] \tag{7.17}$$

also yield a behavior of $1/(t-t_0)$ from arguments similar to those given above. Figure 7 illustrates the trajectories appropriate for carrying out the details. Of course, [1] and [2] can be interchanged.

We can dispose of the terms arising from $g_3^{(0)}(123)$ immediately by noting that at least two binary interactions must have occurred prior to t_0 for these terms to be finite. From solid angle consideration these terms will vanish faster than $1/(t-t_0)$.

We finally consider the terms

$$\int_{t_0}^{t} dt' \; e^{-H^{(0)}(12)(t-t')} \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \Big[\Theta(13) \; e^{-H^{(0)}(123)(t'-t_0)} \, f_1^{(0)}(2) \; g_2^{(0)}(13)$$

$$+ \; \Theta(23) \; e^{-H^{(0)}(123)(t'-t_0)} \; f_1^{(0)}(1) \; g_2^{(0)}(23) \Big] \tag{7.18}$$

and rewrite the first term in (7.18) as

$$\int_{t_0}^{t} dt' \; e^{-H^{(0)}(12)(t-t_0)} \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \; \Theta(13) \; e^{-H^{(0)}(13)(t'-t_0)} \; g_2^{(0)}(13) \, f_1^{(0)}(2)$$

$$+ \int_{t_0}^{t} dt' \left(e^{-H^{(0)}(12)(t-t')} - e^{-H^{(0)}(12)(t-t_0)} \right) \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \; \Theta(13) \; e^{-H^{(0)}(13)(t'-t_0)}$$

$$\times \; g_2^{(0)}(13) \, f_1^{(0)}(2)$$

$$+ \int_{t_0}^{t} dt' \; e^{-H^{(0)}(12)(t-t')} \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \; \Theta(13) \left(e^{-H^{(0)}(123)(t'-t_0)} - e^{-H^{(0)}(13)(t'-t_0)} \right)$$

$$\times \; g_2^{(0)}(13) \; f_1^{(0)}(2) \, . \tag{7.19}$$

The expression

$$G = \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \; \Theta(13) \; e^{-H^{(0)}(13)(t'-t_0)} \; g_2^{(0)}(13) \, f_1^{(0)}(2) \tag{7.20}$$

clearly depends on $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ but not on $\underset{\sim}{x}_{12}$. Then,

$$\left(e^{-H^{(0)}(12)(t-t')} - e^{-H^{(0)}(12)(t-t_0)} \right) \; G(\underset{\sim}{v}_1, \underset{\sim}{v}_2) = 0 \tag{7.21}$$

for $|\underset{\sim}{x}_{12}| < 1$, if both operators project G into the phase space region before a [12] interaction occurs. Thus, (7.21) is true if both

$$t - t' > \frac{1}{|\underset{\sim}{v}_{12}|} , \qquad t - t_0 > \frac{1}{|\underset{\sim}{v}_{12}|} \, . \tag{7.22}$$

We thus conclude that the second line of (7.19) vanishes faster than $1/(t-t_0)^3$. This follows from the argument that (7.22) forces us to consider $t \approx t'$ in order to get a nonzero contribution. Then, substituting t for t' in the $e^{-H^{(0)}(13)(t'-t_0)}$ operator yields

$$|\underset{\sim}{v}_{13}| < \frac{1}{t-t_0} \tag{7.23}$$

in order that $g_2^{(0)}(13)$ is not projected into the region in which it vanishes. We thus find at least a $1/(t-t_0)^3$ decay from the velocity integration alone.

A typical nonvanishing trajectory contributing to the third line of (7.19) is shown in Fig. 8. We see that there must be a [13] interaction prior to t_0 to keep $g_2^{(0)}(13)$ nonzero and there must be a [13] interaction in the interval $t-t_0$ because of $\Theta(13)$. We also require a [12] interaction at time t and a [12] or [32] interaction so that the H operators do not cancel. By comparison with the previous cases, we see that this contribution will certainly decay at least as $1/(t-t_0)$ and most probably faster.

In view of the above, we may write (7.15) as

$$\begin{aligned} f_2^{(1)}(t) = \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \Big(& e^{-H^{(0)}(123)(t-t_0)} - e^{-H^{(0)}(12)(t-t_0)}\, e^{-H^{(0)}(13)(t-t_0)} \\ & - e^{-H^{(0)}(12)(t-t_0)}\, e^{-H^{(0)}(23)(t-t_0)} + e^{-H^{(0)}(12)(t-t_0)} \Big) f_1^{(0)} f_1^{(0)} f_1^{(0)} \\ & + e^{-H^{(0)}(12)(t-t_0)} \Big(f_1^{(0)}(2)\, f_1^{(1)}(t_0, \epsilon t, 1) + f_1^{(0)}(1) f_1^{(1)}(t_0, \epsilon t, 2) + g_2^{(1)}(t_0, \epsilon t) \Big) \\ & + e^{-H^{(0)}(12)(t-t_0)} \int_{t_0}^{t} dt' \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \end{aligned}$$

This equation continued on next page.

$$\Big[\Theta(13) \left(e^{-H^{(0)}(13)(t'-t_0)} - e^{-H^{(0)}(13)(t_\infty-t_0)}\right)\left(f_1^{(0)}(1)\, f_1^{(0)}(3) + g_2^{(0)}(13,t_0)\right) f_1^{(0)}(2)$$

$$+ \Theta(23) \left(e^{-H^{(0)}(23)(t'-t_0)} - e^{-H^{(0)}(23)(t_\infty-t_0)}\right)\left(f_1^{(0)}(2)\, f_1^{(0)}(3) + g_2^{(0)}(23,t_0)\right) f_1^{(0)}(1)\Big]$$

$$+ A(t-t_0) + B(t-t_0) \; . \tag{7.24}$$

Here $A(t-t_0)$ includes all the terms which behave as $1/(t-t_0)$, while B includes those which decay faster than $1/(t-t_0)$.

We then finally get

$$\lim_{t-t_0 \to \infty} \frac{\partial f_1^{(1)}}{\partial \epsilon t} + \frac{\partial f_1^{(0)}}{\partial \epsilon^2 t} = \lim_{t-t_0\to\infty} \int d\underset{\sim}{x}_2 d\underset{\sim}{v}_2\, \Theta(12)\; e^{-H^{(0)}(12)(t-t_0)}$$

$$\times \left(f_1^{(0)}(1)\, f_1^{(1)}(2) + f_1^{(0)}(2)\, f_1^{(1)}(1)\right)$$

$$+ \int d\underset{\sim}{x}_2 d\underset{\sim}{v}_2 d\underset{\sim}{x}_3 d\underset{\sim}{v}_3\; \Theta(12) \Big(e^{-H^{(0)}(123)(t-t_0)} - e^{-H^{(0)}(12)(t-t_0)}\, e^{-H^{(0)}(13)(t-t_0)}$$

$$- e^{-H^{(0)}(12)(t-t_0)}\, e^{-H^{(0)}(23)(t-t_0)} + e^{-H^{(0)}(12)(t-t_0)}\Big)\, f_1^{(0)}(1) f_1^{(0)}(2) f_1^{(0)}(3) \tag{7.25}$$

which is the standard Choh-Uhlenbeck result.

However, we note that in obtaining (7.25) we have dropped the term

$$\lim_{t-t_0\to\infty} e^{-H^{(0)}(12)(t-t_0)}\; g_2^{(1)}(t_0) \tag{7.26}$$

by assuming it to vanish. In fact, we get no information on $g_2^{(1)}(0)$ in this order and, in principle, must determine it by going to next order. We also observe that the term $A(t-t_0)$ produces an error of order ϵ only if we require

$t-t_0 \sim 1/\epsilon$. This also entails computing to higher order to examine the ϵt dependence of the various functions involved. These higher order calculations are not done here but have been carried out in a paper to appear subsequently. The results verify that (7.25) is correct.

We next turn to the time evolution of the system predicted by (7.25). To do this we need some information which follows from the Boltzmann equation (7.1). We know that as $\epsilon t \to \infty$, we have

$$f_1^{(0)} \to n^{(0)} \left(\frac{m}{2\pi\kappa T^{(0)}}\right)^{3/2} \exp - \frac{m}{2\kappa T^{(0)}} \left(\underset{\sim}{v} - \underset{\sim}{V}^{(0)}\right)^2 \tag{7.27}$$

where

$$n^{(0)} = \int d\underset{\sim}{v}\; f_1^{(0)}$$

$$n^{(0)}\underset{\sim}{V}^{(0)} = \int d\underset{\sim}{v}\; \underset{\sim}{v}\; f_1^{(0)}$$

$$\frac{3}{2}\; \frac{n^{(0)}\kappa T^{(0)}}{m} = \frac{1}{2} \int d\underset{\sim}{v} \left(v - V^{(0)}\right)^2 f_1^{(0)}, \tag{7.28}$$

Further, it follows from (7.1) that for all ϵt

$$\frac{\partial n^{(0)}}{\partial \epsilon t} = \frac{\partial \underset{\sim}{V}^{(0)}}{\partial \epsilon t} = \frac{\partial T^{(0)}}{\partial \epsilon t} = 0 \tag{7.29}$$

From (7.25), we have

$$\frac{\partial n^{(1)}}{\partial \epsilon t} + \frac{\partial n^{(0)}}{\partial \epsilon^2 t} = 0$$

$$\frac{\partial}{\partial \epsilon t}\left(n^{(0)}\underset{\sim}{V}^{(1)} + n^{(1)}\underset{\sim}{V}^{(0)}\right) + \frac{\partial}{\partial \epsilon^2 t}\, \underset{\sim}{n}^{(0)} V^{(0)} = 0$$

$$\frac{\partial}{\partial \epsilon t}\left(\frac{3}{2}\, n^{(0)}\kappa T^{(1)} + \frac{3}{2}\, n^{(1)}\kappa T^{(0)}\right) + \frac{\partial}{\partial \epsilon^2 t}\left(\frac{3}{2}\, n^{(0)}\kappa T^{(0)}\right)$$

This equation continued on next page.

$$= - \int d\underset{\sim}{v}_1 d\underset{\sim}{v}_2 d\underset{\sim}{v}_3 d\underset{\sim}{x}_2 d\underset{\sim}{x}_3 \underset{\sim}{v}_1 \cdot \frac{\partial \phi}{\partial \underset{\sim}{x}_{12}} \Big\{ S_{-\infty}(123) - S_{-\infty}(12)\, S_{-\infty}(13)$$

$$- S_{-\infty}(12)\, S_{-\infty}(23) + S_{-\infty}(12) \Big\} f_1^{(0)} f_1^{(0)} f_1^{(0)}$$

$$\equiv Q \quad . \tag{7.30}$$

By secularity removal, (7.30) yields

$$\frac{\partial n^{(1)}}{\partial \epsilon t} = \frac{\partial \underset{\sim}{V}^{(1)}}{\partial \epsilon t} = \frac{\partial n^{(0)}}{\partial \epsilon^2 t} = \frac{\partial \underset{\sim}{V}^{(0)}}{\partial \epsilon^2 t} = \frac{\partial T^{(0)}}{\partial \epsilon^2 t} \tag{7.31}$$

and

$$\frac{3}{2}\, n^{(0)} \kappa \; \frac{\partial T^{(1)}}{\partial \epsilon t} = Q \; . \tag{7.32}$$

We can now return to (7.25) and consider integrating it with respect to ϵt to search for secular behavior. In the limit as $\epsilon t \rightarrow \infty$ we know that

$$\frac{\partial f_1^{(0)}}{\partial \epsilon^2 t} \rightarrow 0 \tag{7.33}$$

since $f_1^{(0)}$ takes on the form (7.27) and (7.31) holds. We also know that the Choh-Uhlenbeck triple collision term vanishes if $f_1^{(0)}$ is Maxwellian. Thus, the only secularity producing term in (7.25) is

$$\int d\underset{\sim}{x}_2\, d\underset{\sim}{v}_2\, \Theta(12)\, S_{-\infty}(12) \Big[f_1^{(0)}(\infty, 1)\, f_1^{(1)}(\infty, \infty, 2)$$

$$+ f_1^{(0)}(\infty, 2)\, f_1^{(1)}(\infty, \infty, 1) \Big] \tag{7.34}$$

which must therefore be set equal to zero. This leads to

$$f_1^{(1)}(\infty, \infty) = f_1^{(0)}(\infty)\; (\alpha + \underset{\sim}{\beta} \cdot \underset{\sim}{v} + \gamma\, v^2) \tag{7.35}$$

which we recognize is simply a perturbed Maxwellian, with α, $\underset{\sim}{\beta}$, and γ determined by normalization. We therefore conclude that if (7.25) has a well-behaved (secularity-free) solution, although we have not found a formal H theorem, the system does tend to thermal equilibrium. It is of interest to note from (7.33) that the system apparently prefers not to evolve on the $\epsilon^2 t$ scale, and that we can, at least to this order, assume that an asymptotic representation is achieved without an $\epsilon^2 t$ dependence.

VIII. CONCLUSIONS

We have seen that it is possible to remove secular behavior to the order we have gone and obtain significant information on the long time, long space behavior of the various functions in so doing. We have also seen that the singular behavior for $|\underset{\sim}{v}_{12}| << 1$ does not influence our results to the order we have gone.

At first glance one might think that some of the above conclusions contradict well known results of thermal equilibrium theory. In fact, upon closer examination one finds that all functions reduce, as they should, to their thermal equilibrium counterparts. It is clear that the long time, long space behavior will be exhibited in thermal equilibrium in examining the fluctuation spectrum.

Finally, we wish to comment on the various terms which appear with the decay $1/t$. This type of behavior has been found by Green and Piccarelli.[15] Such terms will clearly give rise to logarithmic behavior in the next order and thus one should expect $\ln \epsilon$ terms in the expansion. These terms have, in fact, been found[16] and will form the basis for a subsequent paper.

ACKNOWLEDGMENTS

We wish to acknowledge helpful discussions with J. R. Dorfman, R. Guernsey, A. Lenard, and C. H. Su.

This work was performed under the auspices of the Air Force Office of Scientific Research of the Office of Aerospace Research under Contract No. AF 49(638)-1555, and National Aeronautics and Space Administration Grant NSG 220-62.

APPENDIX

We now wish to show that the result of considering the integration limits to be independent of $\underset{\sim}{x}_{13}$ ($\underset{\sim}{x}_{23}$) for I_2 and I_3 in Section III is correct. From (3.17) we have

$$\int_0^t dt' \, S_{-(t-t')}(12) \int d\underset{\sim}{x}_3 \, d\underset{\sim}{v}_3 \; \Theta(13) \, S_{-t'}(123,23) \, f_1^{(0)} f_1^{(0)} f_1^{(0)} \quad . \tag{A.1}$$

In relative coordinates we have

$$\Theta(13) = H^{(0)}(12) - H^{(0)}(123,23) + \underset{\sim}{v}_{13} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{13}} - \underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_{13}} \quad . \tag{A.2}$$

Thus, (A.1) becomes

$$\int_0^t dt' \, S_{-(t-t')}(12) \int d\underset{\sim}{x}_3 d\underset{\sim}{v}_3 \Big(H^{(0)}(12) - H^{(0)}(123,23)$$
$$+ \underset{\sim}{v}_{13} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{13}} - \underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_{13}} \Big) S_{-t'}(123,23) \, f_1^{(0)} f_1^{(0)} f_1^{(0)} \quad . \tag{A.3}$$

We next note that

$$S_{-(t-t')}(12) \, |\underset{\sim}{x}_{13}| = |\underset{\sim}{x}_{13} - \int_{t'}^{t} \underset{\sim}{v}_1(t'') \, dt''| \tag{A.4}$$

where $\underset{\sim}{v}_1$ is obtained from the solution of the two-body problem.

We can then write (A.3) as

$$\int_0^{t} dt' \int_{|\underset{\sim}{x}_{13} - \int_{t'}^{t} \underset{\sim}{v}_1(t'')dt''| = 0}^{|\underset{\sim}{x}_{13} - \int_{t'}^{t} \underset{\sim}{v}_1(t'')dt''| = 1} d\underset{\sim}{x}_3 \, d\underset{\sim}{v}_3 \; S_{-(t-t')}(12) \Big(H^{(0)}(12) - H^{(0)}(123,23) \Big) S_{-t'}(123,23) \, f_1^{(0)} f_1^{(0)} f_1^{(0)}$$

This equation continued on next page.

$$+\int_0^t dt' \int_\alpha^\beta d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3\; S_{-(t-t')}(12) \left(\underset{\sim}{v}_{13} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{13}} - \underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{x}_{13}} \right) S_{-t'}(123, 23)\, f_1^{(0)} f_1^{(0)} f_1^{(0)} \tag{A. 5}$$

where α and β are the same as the limits of the first integral in (A. 5). The first integral in (A. 5) can be written

$$\int_0^t dt' \int_\alpha^\beta d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3\; \frac{\partial}{\partial t'}\; S_{-(t-t')}(12)\; S_{-t'}(123, 23)\; f_1^{(0)} f_1^{(0)} f_1^{(0)} \quad . \tag{A. 6}$$

We further rewrite (A. 6) as

$$\int_0^t dt' \;\frac{\partial}{\partial t'} \int_\alpha^\beta d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3\; S_{-(t-t')}(12)\; S_{-t'}(123, 23)\; f_1^{(0)} f_1^{(0)} f_1^{(0)}$$

$$-\int_0^t dt' \;\frac{\partial}{\partial t'''} \left[\int_{|\underset{\sim}{x}_{13} - \int_{t'''}^{t} \underset{\sim}{v}_1 (t'')\, dt''| = 0}^{|\underset{\sim}{x}_{13} - \int_{t'''}^{t} \underset{\sim}{v}_1 (t'')\, dt''| = 1} d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3\; S_{-(t-t')}(12)\; S_{-t'}(123, 23)\; f_1^{(0)} f_1^{(0)} f_1^{(0)} \right]_{t''' = t'} . \tag{A. 7}$$

Our object now is to show that the last term in (A. 5) cancels the second term in (A. 7) leaving a finite term plus the secularity producing term that we have already obtained in I_2. Thus, upon using Gauss' theorem, the last term in (A. 5) becomes

$$\int_0^t dt' \int_\sigma \; S_{-(t-t')}(12)\; S_{-t'}(123, 23)\; \underset{\sim}{v}_1 (t') \cdot d\underset{\sim}{\sigma}\; d\underset{\sim}{v}_3\; f_1^{(0)} f_1^{(0)} f_1^{(0)} \tag{A. 8}$$

where σ is the surface of the sphere shown in Fig. A. 1. To treat the second integral in (A. 7) it is convenient to write the limits in explicit rather than implicit form. Thus we get

$$\underset{\sim}{x}_{13} = \int_{t'''}^{t} \underset{\sim}{v}_1(t'')\,dt'' \tag{A.9}$$

for the lower limit and

$$\underset{\sim}{x}_{13} = \underset{\sim}{1}(\sigma) + \int_{t'''}^{t} \underset{\sim}{v}_1(t'')\,dt'' \tag{A.10}$$

for the upper limit. We therefore write

$$-\int_0^t dt' \frac{\partial}{\partial t'''} \left[\int dx_3\, dv_3\, S_{-(t-t')}(12)\, S_{-t'}(123,23)\, f_1^{(0)} f_1^{(0)} f_1^{(0)} \right]_{\underset{\sim}{x}_{13} = +\int_{t'''}^{t} \underset{\sim}{v}_1(t'')\,dt''}^{\underset{\sim}{x}_{13} = \underset{\sim}{1}(\sigma) - \int_t^{t'''} \underset{\sim}{v}_1(t'')\,dt''}$$

$$= + \int_0^t dt' \int_\sigma S_{-(t-t')}(12)\, S_{-t'}(123,23)\, \underset{\sim}{v}_1(t') \cdot d\underset{\sim}{\sigma}\, d\underset{\sim}{v}_3\, f_1^{(0)} f_1^{(0)} f_1^{(0)}\,. \tag{A.11}$$

The last result follows from simply carrying out the time derivatives using

$$\frac{\partial}{\partial t'''} = \sum_{j=1}^{3} \frac{\partial}{\partial x_{13_j}} \frac{\partial x_{13_j}}{\partial t'''}\,. \tag{A.12}$$

Thus, (A.1) becomes

$$\int_0^t dt' \frac{\partial}{\partial t'} \int_\alpha^\beta d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3\, S_{-(t-t')}(12)\, S_{-t'}(123,23)\, f_1^{(0)} f_1^{(0)} f_1^{(0)}$$

$$+ \int_0^t dt' \int_\alpha^\beta d\underset{\sim}{x}_3\, d\underset{\sim}{v}_3\, S_{-(t-t')}(12)\, \underset{\sim}{v}_{13} \cdot \frac{\partial}{\partial \underset{\sim}{x}_{13}}\, S_{-t'}(123,23)\, f_1^{(0)} f_1^{(0)} f_1^{(0)}$$

(A.13)

which essentially agrees with (3.22) since the $H_r(13)\,t_\infty$ terms can be treated in a similar fashion to those above and the differences between the H and H_r operators vanish. It is clear that arguments similar to those used above can also be used on (3.30) in examining I_3.

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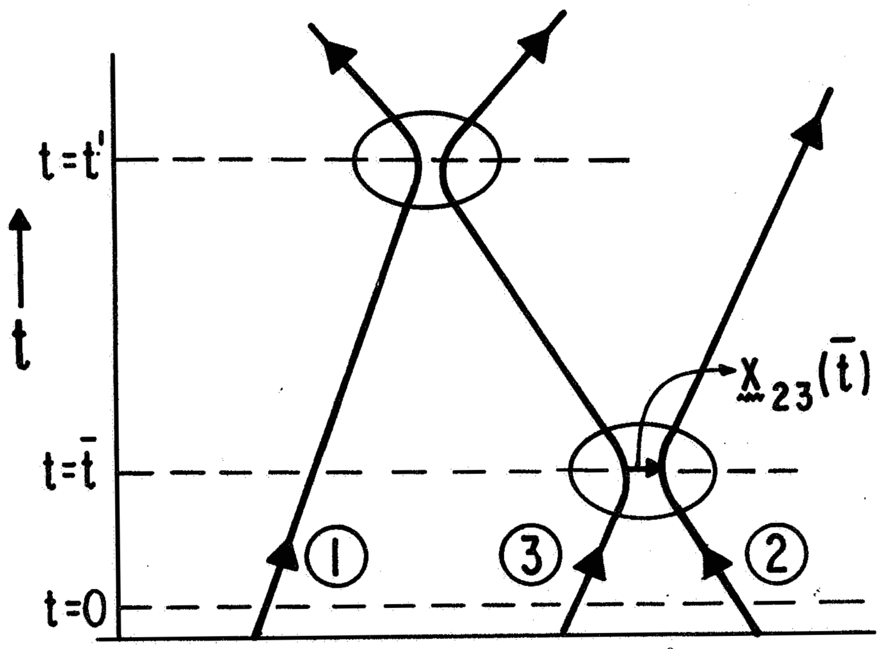

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Fig. 1 The [23] - [13] trajectories.

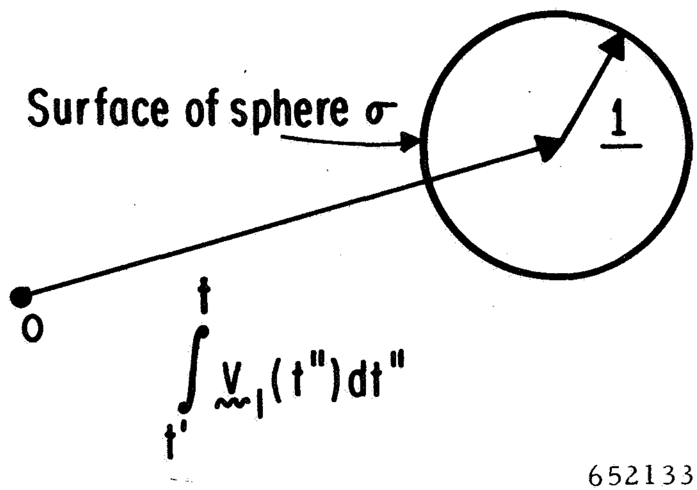

Fig. A.1. Sphere of integration for (A.8).

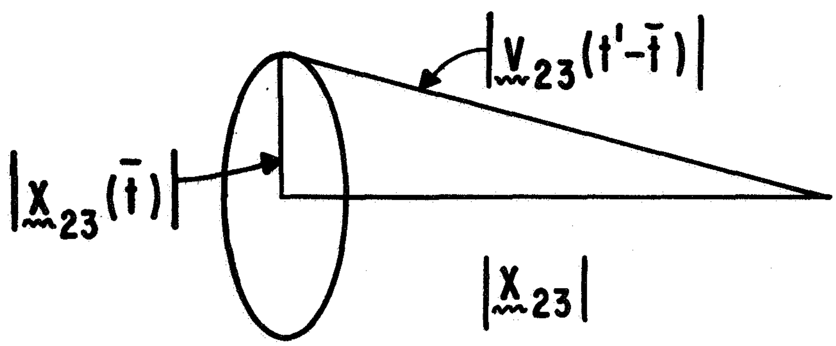

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Fig. 2. Cone of allowed velocities.

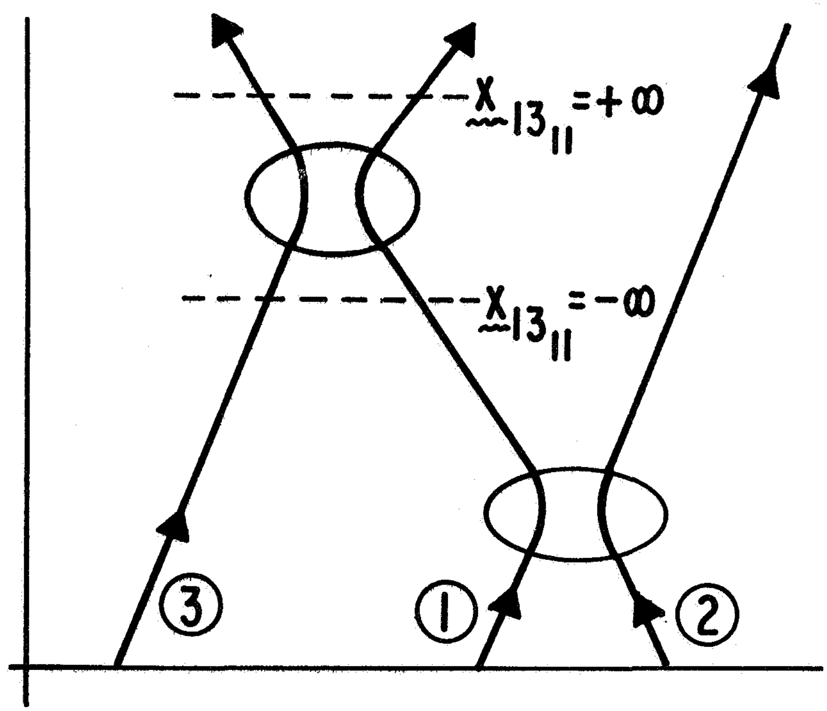

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Fig. 3. Trajectories for (3.23).

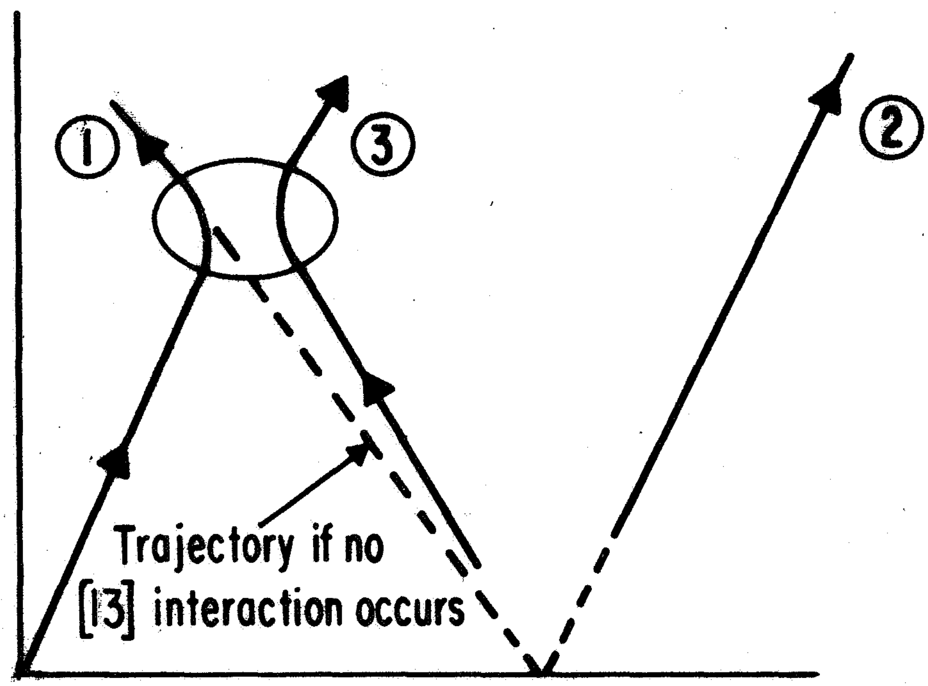

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Fig. 4. Trajectories for $\underset{\sim}{x}_{13_{\parallel}} = +\infty$ in (3.32).

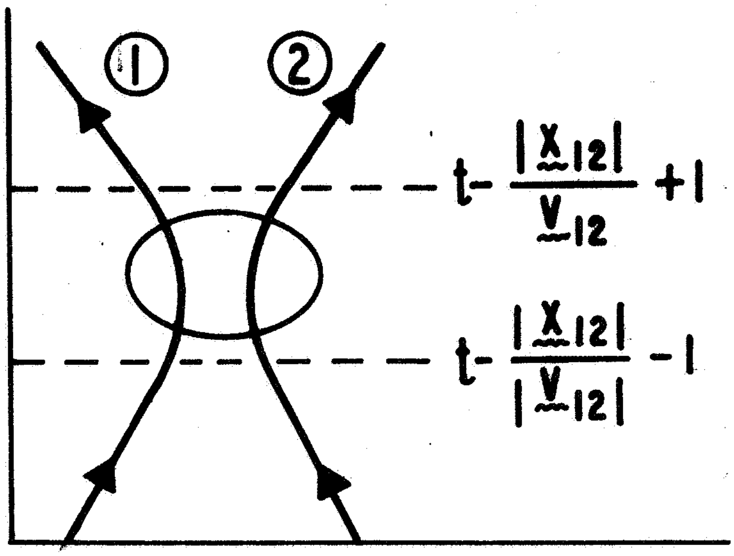

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Fig. 5. Trajectories for (4.2).

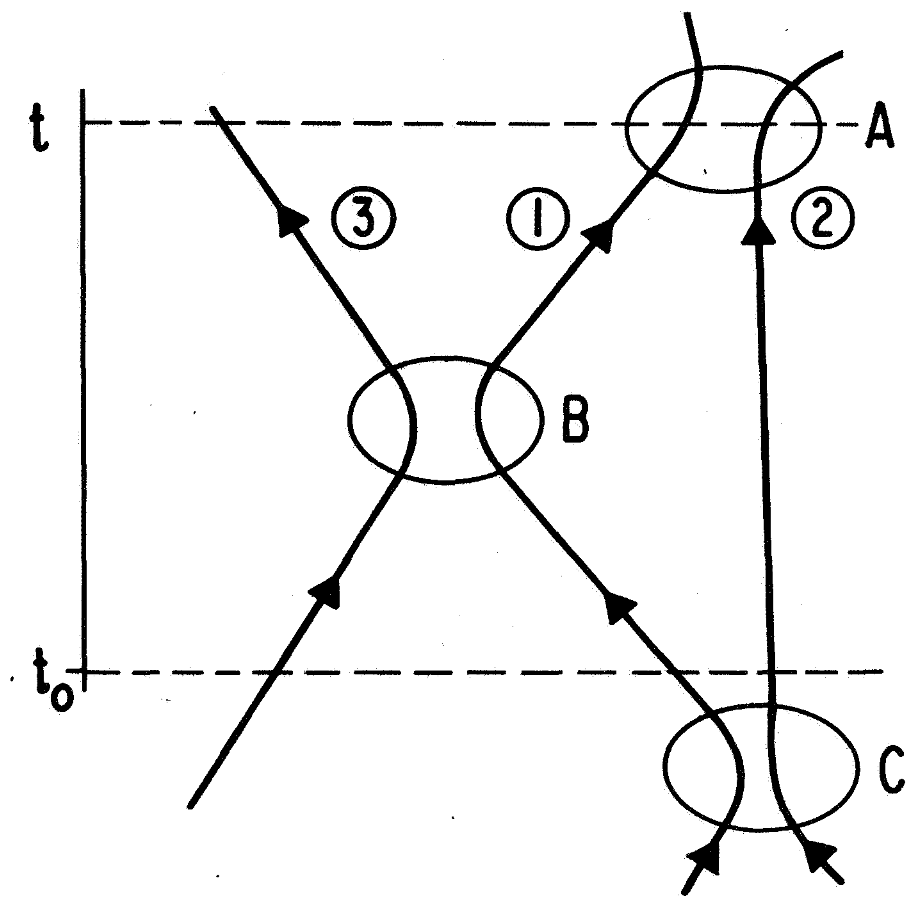

Fig. 6. Trajectories for (7.16).

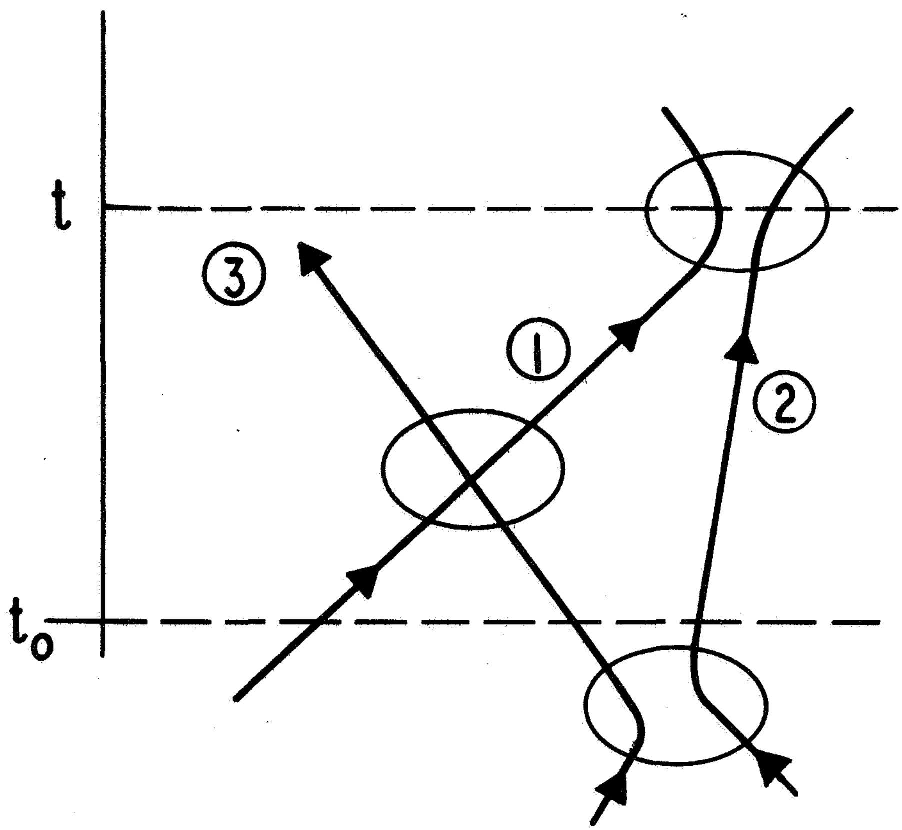

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Fig. 7. Trajectories for (7.17).

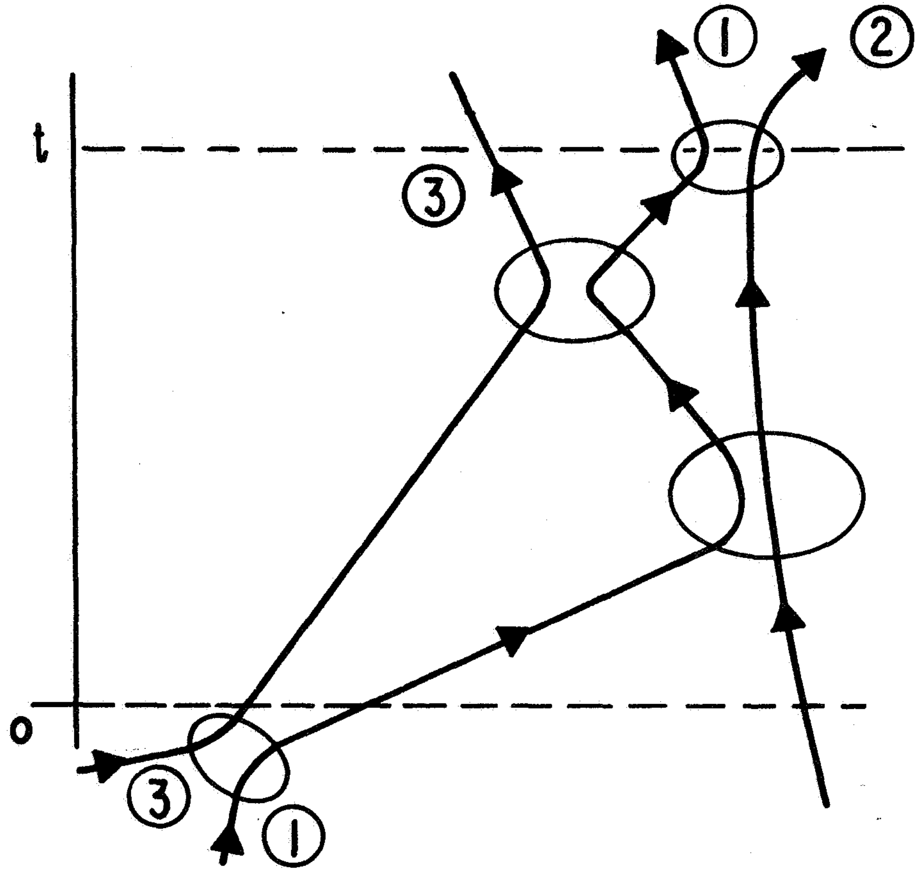

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Fig. 8. Trajectories for the third line in (7.19).